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Fractal dimension for fractal structures: A Hausdorff approach revisited

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ABSTRACT

In this paper, we use fractal structures to study a new approach to the Hausdorff dimension from both continuous and discrete points of view. We show that it is possible to generalize the Hausdorff dimension in the context of Euclidean spaces equipped with their natural fractal structure. To do this, we provide three definitions of fractal dimension for a fractal structure and study their relationships and mathematical properties.

One of these definitions is in terms of finite coverings by elements of the fractal structure. We prove that this dimension is equal to the Hausdorff dimension for compact subsets of Euclidean spaces. This may be the key for the creation of new algorithms to calculate the Hausdorff dimension of these kinds of space.

1. Introduction

The importance of fractal patterns in science is supported by a great number of applications to diverse research areas where fractals have been identified in recent years. In this way, one of the main tools that have been used to study fractals is the fractal dimension, usually understood as the classical *box-counting* or Hausdorff dimension, since it is the basic invariant of a fractal set that provides useful information about its complexity. Thus, note that fractal dimension theory has been widely used in physical applications such as the study of dynamical systems [7], strange attractors [11], cosmology [4], geophysics [12], and quantum theory [15], to name just a few.

The box-counting dimension is more useful for practical applications whereas the Hausdorff dimension presents "better" analytical properties, since its definition is based on a measure. Nevertheless, the Hausdorff dimension can be difficult or even impossible to estimate in empirical applications. Accordingly, though these models to determine the fractal dimension of a set may be defined for any metrizable space, most of the empirical applications of fractal dimension are considered in the context of Euclidean spaces with box-counting dimension. However, the Hausdorff dimension is still used to classify spaces. For instance, in [14], the authors apply the Hausdorff dimension to classify Moran fractals by a quasi-Lipschitz equivalence. In addition to that, note that, in recent years, some effort has been made in order to extend the use of the box-counting dimension in higher-dimensional Euclidean spaces (see [20]) and to provide new definitions of fractal dimension based on probability measures by means of variational principles (see [13]).

The introduction of fractal structures is very suitable for studying fractals from the point of view of asymmetric topology. In particular, fractal structures have appeared in some contexts where their use is natural, such as transitive quasi-uniformities, metrization, space-filling curves, topological and fractal dimensions, and self-similar sets. See [16,17] for a detailed description of them.

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It turns out that the concept of fractal structure is especially suitable to provide a definition of fractal dimension. In this way, as has been shown in some recent papers (see [8,9]), it is possible to define new concepts of fractal dimension to calculate this quantity for any space with respect to a fractal structure. Thus, in a first attempt, we provided two models of fractal dimension for a fractal structure that generalized the classical box-counting dimension in the context of Euclidean spaces equipped with their natural fractal structure. In addition to that, they allowed us to calculate the fractal dimension in non-Euclidean contexts such as the domain of words (see [10]). On the other hand, in [8, Definition 4.2], we introduced the so-called fractal dimension III, which was a new model of fractal dimension based on an appropriate discretization of both the Hausdorff measure and the Hausdorff dimension. This has been successfully applied to financial markets [18].

This paper has a double purpose. First, we look for new models of fractal dimension for a fractal structure that could allow us to approach the Hausdorff dimension in the context of Euclidean spaces, and even generalize it. Additionally, since in some research papers a way to calculate the Hausdorff dimension of some kind of sets has been investigated (see [19] for instance), we also look for a definition of fractal dimension for a fractal structure and a theoretical result that could allow us to computationally calculate the Hausdorff dimension of a compact Euclidean subset.

Thus, the organization of the paper is as follows. In Section 2, we include some concepts, notation, and results about fractal structures and fractal dimensions. In Section 3.1, we formally define fractal dimensions IV, V, and VI for a fractal structure and show some results that relate these definitions to each other, and also to the classical fractal dimensions and the so-called fractal dimensions I, II, and III (that were introduced and studied in detail in previous papers). In Section 3.3, we explore some specific analytical properties of fractal dimension VI, and show that its definition is based on a measure, as it happens with Hausdorff dimension. In addition, we present our main theorems in Section 3.4. First, we show in Theorem 3.10 and Corollary 3.11 that both fractal dimensions V and VI generalize the Hausdorff dimension in the Euclidean context, and then we prove in Theorem 3.13 that the fractal dimension IV of any bounded Euclidean subset is equal to the Hausdorff dimension of its closure. This is the theoretical result we could apply in order to provide a computational approach to the Hausdorff dimension of Euclidean compact subsets. Thus, fractal dimension IV gives an intermediate model between box-counting and Hausdorff dimensions that could be very useful in future applications.

2. Preliminaries

The main purpose of this section is to recall some necessary notation, concepts, and results that will be helpful in this paper. In this way, we will focus on fractal structures, box-counting and fractal dimensions I and II models, and Hausdorff dimension and fractal dimension III definitions.

2.1. Fractal structures

Fractal structures constitute the main concept we use in this paper to develop a new theory about fractal dimension. The concept of fractal structure was first introduced in [1] from a topological point of view to characterize non-Archimedeanly quasi-metrizable spaces, but it can also be used to study fractals. For example, in [3] it was used to study attractors of iterated function systems.

Let Γ_1 and Γ_2 be two coverings of X. We will write $\Gamma_1 \prec \Gamma_2$ to denote that Γ_1 is a *refinement* of Γ_2, namely, for all $A \in \Gamma_1$, there exists $B \in \Gamma_2$ such that $A \subseteq B$. In addition, the notation $\Gamma_1 \prec\prec \Gamma_2$ means that $\Gamma_1 \prec \Gamma_2$, and, for all $B \in \Gamma_2$, we have that $B = \bigcup\{A \in \Gamma_1 : A \subseteq B\}$.

Definition 2.1. A fractal structure on a set X is a countable family of coverings of X, $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$, such that $\Gamma_{n+1} \prec\prec \Gamma_n$, for all $n \in \mathbb{N}$. $(X, \mathbf{\Gamma})$ is called a generalized fractal space or simply a GF-space.

The topology induced by the fractal structure on X is given by the neighborhood base $\mathcal{U}_x^{\mathbf{\Gamma}} = \{U_{xn} : n \in \mathbb{N}\}$ for each $x \in X$, where $U_{xn} = X \setminus \bigcup\{A \in \Gamma_n : x \notin A\}$.

The coverings of $\mathbf{\Gamma}$ are called levels, so Γ_n is level n of fractal structure $\mathbf{\Gamma}$.

A fractal structure $\mathbf{\Gamma}$ is said to be finite if all levels Γ_n are finite coverings.

Note that any level of a fractal structure is a closure-preserving closed covering (see [2, Proposition 2.4]).

2.2. Box-counting dimension and fractal dimensions I and II

The popularity of the box-counting dimension is mainly due to the possibility of its effective calculation and empirical estimation in Euclidean contexts. The basic theory of the box-counting dimension can be found in [6]. Next, we recall the definition of the standard box-counting dimension.

Definition 2.2. The (lower/upper) box-counting dimension of a subset $F \subseteq \mathbb{R}^d$ is given by the following (lower/upper) limit:

$$\dim_B(F) = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta},$$

where δ is the scale and $N_\delta(F)$ can be calculated as the number of δ-cubes that intersect F. See [6, Equivalent Definitions 3.1] for other equivalent ways to calculate $N_\delta(F)$.