J. Math. Anal. Appl. 409 (2014) 331–346

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Spectrality of one dimensional self-similar measures with consecutive digits

Qi-Rong Deng

Department of Mathematics, Fujian Normal University, Fuzhou, 350007, PR China

ARTICLE INFO

Article history:
Received 21 May 2013
Available online 24 July 2013
Submitted by D. Waterman

Keywords:
Spectral measure
Self-similar measure
Consecutive digits

ABSTRACT

Assume $0 < |\rho| < 1$ and m is a prime, let μ_ρ be the self-similar measure defined by $\mu_\rho(A) = \frac{1}{m}\sum_{j=0}^{m-1}\mu_\rho(\rho^{-1}A - j), \forall A \in \mathcal{B}$. We prove that $L^2(\mu_\rho)$ contains an orthonormal basis of exponential functions if and only if $\rho = \pm 1/mk$ for some $k \in \mathbb{N}$.

1. Introduction

Let μ be a Borel probability measure on $\mathbb{R}$. We say that μ is a spectral measure if there exists a discrete set Λ such that $E_\Lambda := \{e^{2\pi\lambda xi} : \lambda \in \Lambda\}$ forms an orthonormal basis of $L^2(\mu)$. In this case, we call Λ a spectrum of μ and (μ, Λ) a spectral pair, respectively.

Jorgenson and Pederson [4]studied the spectral property of general Cantor measures. They proved that the $1/k$-Cantor measure $\mu_{1/k}$ on $\mathbb{R}$ is a spectral measure if k is even (Strichartz provided a simplified proof in [9]). This result was investigated by Laba and Wang in more details in [5] and for the general Borel measures in [6].

Hu and Lau [3] further studied the spectral property of Bernoulli convolutions. They proved that the necessary and sufficient condition that the Bernoulli convolution has an infinite orthonormal set E_Λ of exponential functions is that the contraction ratio ρ is the n-th root of a fraction p/q, where p is odd and q is even. Recently, Dai [1] proved that the Bernoulli convolution has an orthonormal basis E_Λ of exponential functions if and only if the contraction ratio ρ is the reciprocal of an even integer.

Motivated by the above results, we study the spectral property of one dimensional self-similar measures with consecutive digits.

Let ρ be a real number such that $0 < |\rho| < 1$, it is well known that for any positive integer $m \geq 2$, there exists a unique probability measure, denoted by μ_ρ, such that

$$\mu_\rho(A) = \frac{1}{m}\sum_{j=0}^{m-1}\mu_\rho(\rho^{-1}(A) - j) \tag{1.1}$$

for all Borel set $A \in \mathcal{B}$. μ_ρ is called a self-similar measure.

For the self-similar measure μ_ρ defined in (1.1), our main theorem is as follows.

E-mail addresses: qrdeng@fjnu.edu.cn, dengfractal@126.com.

0022-247X/$ – see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.jmaa.2013.07.046

Theorem 1.1. *If m is a prime, then $L^2(\mu_\rho)$ contains an orthonormal basis of exponential functions only if $\rho = \pm\frac{1}{mk}$ for some $k \in \mathbb{N}$.*

This is an extension of the result of [1].

On the other hand, Dai etc. proved that $L^2(\mu_{1/mk})$ contains an orthonormal basis of exponential functions for any $k \in \mathbb{N}$ in [2]. It is easy to see that $L^2(\mu_{-1/mk})$ also contains an orthonormal basis of exponential functions for any $k \in \mathbb{N}$. Hence we have the following.

Theorem A. *If m is a prime, then $L^2(\mu_\rho)$ contains an orthonormal basis of exponential functions if and only if $\rho = \pm\frac{1}{mk}$ for some $k \in \mathbb{N}$.*

Remarks. Theorem 1.1 indicates that the main theorems of [3,1] also hold for $-1 < \rho < 0$. Our proof of Theorem 1.1 strongly depends on the structure of the zeros of $\hat{\mu}_\rho$. The set of zeros of $\hat{\mu}_\rho$ will be very complicated if the digit set is replaced by a non-consecutive digit set. So far, we do not know how to deal with the case of non-consecutive digits. Also, some of our proofs do not work when m is not a prime. For integral self-affine measures, Li studied the spectrality of a class of planar self-affine measures with decomposable digit sets in [7] and with three non-consecutive digit set in [8].

If we only consider the existence of infinite orthonormal set of exponential functions, we have the following theorem which is an extension of the result of [3].

Theorem 1.2. *Assume m is a prime, then $L^2(\mu_\rho)$ contains an infinite orthonormal set of exponential functions if and only if $\rho = \pm(q/p)^{1/r}$ for some $p, q, r \in \mathbb{N}$ with the properties: p, q are co-prime and $m|p$.*

Since E_Λ forms an orthonormal set in $L^2(\mu_\rho)$ if and only if $E_{t+\Lambda}$ forms an orthonormal set in $L^2(\mu_\rho)$ for any fixed $t \in \mathbb{R}^d$. For simplicity we assume that $0 \in \Lambda$ throughout this paper.

Notations. We will use the following notations. Let $\mathbb{Z}$ be the set of all integers, let $\mathbb{N}$ be the set of all positive integers. For any $\mathbf{x}, \mathbf{y} \in \mathbb{Z}$ and $r \in \mathbb{N}$, we use $\mathbf{x} \equiv \mathbf{y} (\mathrm{mod}\ r)$ to denote $\mathbf{x} - \mathbf{y} \in r\mathbb{Z}$.

For the iterated function system $\{S_j\}_{j=0}^{m-1}$ with $S_j(x) = \rho(x+j)$ and the associated μ_ρ defined in (1.1), let $\hat{\mu}_\rho(t) = \int e^{2\pi xti} d\mu_\rho(x)$ be the Fourier transform of μ_ρ. Define

$$\mathcal{Z}_\rho = \{t \in \mathbb{R} : \hat{\mu}_\rho(t) = 0\}$$

to be the set of zeros of $\hat{\mu}_\rho(t)$. Let

$$\mathbb{O} = \{\pm(q/p)^{1/r} : p, q, r \in \mathbb{N}\}.$$

It is clear that $\beta \in \mathbb{O}$ if and only if $|\beta|$ is an algebraic rational with a minimal polynomial $px^r - q$ for some $p, q, r \in \mathbb{N}$.

Throughout this paper, we always use

$$E_\Lambda = \{e^{2\pi\lambda xi} : \lambda \in \Lambda\},$$

to denote an orthonormal set of exponential functions in $L^2(\mu_\rho)$, where Λ is a subset of $\mathbb{R}$ containing 0. For this Λ, we define

$$Q_\Lambda(t) = \sum_{\lambda\in\Lambda} |\hat{\mu}_\rho(\lambda - t)|^2.$$

We organize the paper as follows. Some preliminary lemmas are given in Section 2. Section 3 is devoted to prove Theorem 1.2. While Theorem 1.1 is proven in Section 4.

2. Some preliminary lemmas

We first give some preliminary results associated with the self-similar measure μ_ρ. Then we will use them to prove Theorems 1.1 and 1.2 in Sections 3 and 4, respectively.

It is easy to prove the following.

Lemma 2.1. *Let $\hat{\mu}_\rho(t)$ be the Fourier transform of the self-similar measure μ_ρ defined in (1.1), then*

$$\hat{\mu}_\rho(t) = \prod_{k=1}^{n} \left[\frac{1}{m}\sum_{j=0}^{m-1}(e^{2\pi\rho^k ti})^j\right] \hat{\mu}_\rho(\rho^n t) \tag{2.1}$$

for all positive integers $n > 0$.

Lemma 2.2. $\mathcal{Z}_\rho = \{\frac{\ell}{m\rho^k} : k \in \mathbb{N}, \ell \in \mathbb{Z}\setminus m\mathbb{Z}\}$.