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Stability of the Riemann solutions for a Chaplygin gas

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ABSTRACT

In this paper, we study the structural stability of solutions to the Riemann problem for one-dimensional isentropic Chaplygin gas. We perturb the Riemann initial data by taking three piecewise constant states and construct the global structure. By letting the perturbed parameter ε tend to zero, we prove that the Riemann solutions are stable under the local small perturbations of the Riemann initial data even when the initial perturbed density depending on the parameter but with no mass concentration limit.

1. Introduction

In this paper, we are concerned with the one-dimensional isentropic Chaplygin gas dynamics characterized by

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p)_x = 0 \end{cases} \tag{1.1}$$

with state equation $p = -\frac{1}{\rho}$, where $\rho > 0$ is the density, u is the velocity. One remarkable feature of this dynamics is that the characteristic fields are linearly degenerate. For the physical meaning of Chaplygin gas, we refer readers to [1–3,9].

Recently, there are many works on the existence of solution to the one-dimensional Chaplygin gas. The existence of classical solution to the Cauchy problem has been thoroughly studied in [8]. The research on the existence of solution including delta waves is still mainly concentrated on Riemann problems. For the isentropic case, Brenier found that it owns solution with concentration [2]. Guo et al. constructed the Riemann solution with delta wave [6]. Wang et al. proved the existence of the Riemann problem with delta initial data [13]. For the adiabatic case, Sheng et al. constructed the Riemann solution in [14]. The Riemann problem for one dimensional generalized Chaplygin gas dynamics is studied in [12], the characteristic fields of the dynamics in which are genuinely nonlinear. However, there are few works on the stability of the solution to one-dimensional isentropic Chaplygin gas, even on that of the Riemann solution.

Consider the Riemann problem of (1.1) with initial data

$$(\rho, u)(x, 0) = \begin{cases} (\rho_l, u_l), & x < 0 \\ (\rho_r, u_r), & x > 0, \end{cases} \tag{1.2}$$

where $\rho_{l,r}$, $u_{l,r}$ are all given constants.

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In order to investigate the stability of Riemann solution for the Chaplygin gas, we consider the initial data consisting of three piecewise constant states as follows:

$$(\rho, u)(x, 0) = \begin{cases} (\rho_l, u_l), & x < -\epsilon \\ (\rho_m, u_m), & -\epsilon < x < \epsilon \\ (\rho_r, u_r), & x > \epsilon, \end{cases} \tag{1.3}$$

where $\varepsilon > 0$ is arbitrarily small. If $\lim_{\varepsilon\to 0} \varepsilon\rho_m = 0$, then by passing to the limit as $\varepsilon \to 0$ in (1.3) we get the corresponding Riemann initial data (1.2), thus the initial data (1.3) can be viewed as a perturbation of the corresponding Riemann initial data (1.2). This method has been used by many authors to study the stabilities of Riemann solution [10,11] and even to construct the unique solution of Riemann solution [7,13].

For the Riemann problem of (1.1), a delta wave may be present in the solution. When constructing the global solution of (1.1) and (1.3), one needs to consider whether two adjacent waves intersect and interact with each other. It is not so easy to see whether two delta waves meet and how they interact with each other for the model considered here and thus needs some technical treatment. Since (1.1) and (1.2) are scaling invariant, people usually construct its self-similar solutions. By studying the limit structure of the solution of (1.1) and (1.3), we prove that the Riemann solutions of (1.1) and (1.2) are stable under this local small perturbation of the Riemann initial data by letting $\varepsilon \to 0$.

This paper is organized as follows. In Section 2 we recall and present some known results about the system (1.1) and its Riemann solution with initial data (1.2). In Section 3 we mainly discuss the interaction of the elementary waves, namely, shock waves, rarefaction waves and delta waves, for all the cases when the initial data are in the form of (1.3) with u_m and ρ_m being quantities independent of the parameter ε. By letting $\varepsilon \to 0$ we get the limit solution and by comparing it with the corresponding Riemann problem of (1.1) and (1.2), we find that the Riemann solution is stable under this perturbation. In Section 4, we investigate the case that ρ_m is a function of ε satisfying $\lim_{\varepsilon\to 0} \rho_m = \infty$ and $\lim_{\varepsilon\to 0} \varepsilon\rho_m = 0$. By studying the limit structure of the perturbed solution we find that the Riemann problem is also stable under these local perturbation. A conclusion of this paper is given in Section 5.

2. Riemann problem for (1.1) and (1.2)

The solution of (1.1) and (1.2) is referred to [5,6,13]. For the readers' convenience we recall and present some facts as follows.

The two eigenvalues of (1.1) are

$$\lambda_1 = u - c \quad \text{and} \quad \lambda_2 = u + c,$$

where $c = \frac{1}{\rho}$ is the sound speed. The corresponding eigenvectors are

$$\vec{r}_1 = (\rho, -c)^t \quad \text{and} \quad \vec{r}_2 = (\rho, c)^t.$$

System (1.1) is strictly hyperbolic for $\rho > 0$ and the fact $\nabla_{(\rho,u)}\lambda_i \cdot \vec{r}_i \equiv 0,\ i = 1, 2$ implies that λ_i $(i = 1, 2)$ are linearly degenerate.

Since (1.1) and (1.2) are invariant under scaling $(x, t) \mapsto (\tau x, \tau t)$, we seek the self-similar solution

$$(\rho, u)(x, t) = (\rho, u)(\xi), \quad \xi = \frac{x}{t}.$$

Then (1.1) and (1.2) are reduced to the following boundary value problem of ordinary differential equations

$$\begin{cases} -\xi\rho_\xi + (\rho u)_\xi = 0 \\ -\xi(\rho u)_\xi + \left(\rho u^2 - \dfrac{1}{\rho}\right)_\xi = 0 \end{cases} \tag{2.1}$$

$$(\rho, u)(-\infty) = (\rho_l, u_l) \qquad (\rho, u)(+\infty) = (\rho_r, u_r).$$

In addition to the solution $(\rho, u) = const.$, (2.1) has solution

$$\xi = u \pm c = u_l \pm c_l.$$

For a bounded discontinuity at $\xi = \sigma$, the Rankine–Hugoniot conditions hold

$$\begin{cases} \sigma[\rho] = [\rho u], \\ \sigma[\rho u] = [\rho u^2 + p]. \end{cases} \tag{2.2}$$

Hereafter, $[\rho] = \rho_r - \rho_l$, where $\rho_{l,r}$ are the value of ρ on the left and right sides of the discontinuity, etc. It follows from (2.2) that

$$\sigma = u \pm c = u_l \pm c_l.$$