J. Math. Anal. Appl. 409 (2014) 393–398

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Compactness of the $\overline{\partial}$-Neumann operator and commutators of the Bergman projection with continuous functions

Mehmet Çelik [a], Sönmez Şahutoğlu [b,*]

[a] *University of North Texas at Dallas, Department of Mathematics & Information Sciences, 7400 University Hills Blvd., Dallas, TX 75241, USA*

[b] *University of Toledo, Department of Mathematics & Statistics, 2801 W. Bancroft, Toledo, OH 43606, USA*

ARTICLE INFO

Article history:
Received 23 March 2013
Available online 16 July 2013
Submitted by Richard M. Timoney

Keywords:
$\overline{\partial}$-Neumann operator
Hankel operators
Bergman projection
Pseudoconvex domain

ABSTRACT

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, $0 \leq p \leq n$, and $1 \leq q \leq n-1$. We show that compactness of the $\overline{\partial}$-Neumann operator, $N_{p,q+1}$, on square integrable $(p, q+1)$-forms is equivalent to compactness of the commutators $[P_{p,q}, \overline{z}_j]$ on square integrable $\overline{\partial}$-closed (p, q)-forms for $1 \leq j \leq n$ where $P_{p,q}$ is the Bergman projection on (p, q)-forms. We also show that compactness of the commutator of the Bergman projection with bounded functions percolates up in the $\overline{\partial}$-complex on $\overline{\partial}$-closed forms and square integrable holomorphic forms.

Published by Elsevier Inc.

The purpose of this paper is to characterize compactness of the $\overline{\partial}$-Neumann operator on square integrable (p, q)-forms. Theorem 1 provides six equivalent statements, for q at least 2, on bounded pseudoconvex domains. However, the important special case of functions, namely $(0, 0)$-forms, remains open. In Remark 3 we discuss why our proof breaks down in this case.

Compactness results in the $\overline{\partial}$-Neumann problem have a long history; we refer the reader to a recent book by Straube [12] for a detailed discussion. We note here that most known results provide conditions for compactness in terms of the boundary geometry. It is also useful to characterize compactness in functional analytic terms. For example, Catlin and D'Angelo [1, Theorem 1] used compactness of the commutators $[P, \phi]$ of the Bergman projection P and certain multiplication operators ϕ in conjunction with a complex variables analogue of Hilbert's 17th problem (see also [5]). In the same paper they showed that compactness of $N_{0,1}$ implies that the commutators $[P, M]$ are compact for all tangential pseudodifferential operators M of order 0. D'Angelo then asked the following question:

Question 1. *Can one characterize compactness of the $\overline{\partial}$-Neumann operator in terms of commutators $[P, \phi]$?*

This question is appealing because of its connection to operator theory as well. Let $A^2(\Omega)$ be the space of square integrable holomorphic functions on Ω. The Hankel operator $H_\phi : A^2(\Omega) \to L^2(\Omega)$, with a bounded symbol ϕ, is defined by $H_\phi(f) = (I-P)(\phi f)$. Using Kohn's formula, $P = I - \overline{\partial}^* N_{0,1} \overline{\partial}$, one obtains that $H_\phi f = \overline{\partial}^* N_{0,1} \overline{\partial}(f\phi)$. Using this formula, Čučković and Şahutoğlu [4] studied how boundary geometry interacts with Hankel operators. They showed that, on smooth bounded convex domains in $\mathbb{C}^2$, compactness of H_ϕ can be characterized by the behaviour of ϕ on analytic discs in the boundary. Here ϕ is smooth up to the boundary. It would be interesting to know whether this characterization still holds in higher dimensions.

* Corresponding author.
E-mail addresses: mehmet.celik@unt.edu (M. Çelik), sonmez.sahutoglu@utoledo.edu (S. Şahutoğlu).

0022-247X/$ – see front matter. Published by Elsevier Inc.
http://dx.doi.org/10.1016/j.jmaa.2013.07.015

On convex domains the relation between the compactness of the commutators and that of the $\overline{\partial}$-Neumann operator has been fairly well understood: if Ω is a bounded convex domain, then compactness of $N_{p,q+1}$ is equivalent to compactness of the commutators $[P_{p,q}, \phi]$ on the space of (p, q)-forms with square integrable holomorphic coefficients, for all functions ϕ continuous on $\overline{\Omega}$ (see [12, Remark (ii) in Section 4.1]).

Theorem 1 fails on non-pseudoconvex domains. In our previous paper [2] we constructed a smooth bounded non-pseudoconvex domain in $\mathbb{C}^n$ (for a given $n \geq 3$) for which the commutators $[P, \phi]$ are compact (on square integrable functions) for all ϕ continuous on the closure of the domain, yet the $\overline{\partial}$-Neumann operator $N_{0,1}$ is not compact. In this paper we will consider the important pseudoconvex case and establish a decisive result on forms in Theorem 1.

1. Background and the main results

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. We denote the square integrable (p, q)-forms on Ω by $L^2_{(p,q)}(\Omega)$ and the subspace of $\overline{\partial}$-closed forms by $K^2_{(p,q)}(\Omega)$. The $\overline{\partial}$-Neumann operator, $N_{p,q}$, is defined as the solution operator for $\Box_{p,q}u = v$ where $\Box_{p,q} = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ on $L^2_{(p,q)}(\Omega)$ and $\overline{\partial}^*$ is the Hilbert space adjoint of $\overline{\partial}$. Hörmander [8] showed that $N_{p,q}$ is a bounded operator when Ω is a bounded pseudoconvex domain. Kohn in [9] connected the Bergman projection $P_{p,q}$ on (p, q)-forms with the $\overline{\partial}$-Neumann operator through the formula

$$P_{p,q} = I - \overline{\partial}^* N_{p,q+1}\overline{\partial}.$$

We note that $P_{p,q}$ is the orthogonal projection onto $K^2_{(p,q)}(\Omega)$, the operator $P_{0,0}$ is the classical Bergman projection P, and $K^2_{(0,0)}(\Omega)$ (also denoted as $A^2(\Omega)$) is called the Bergman space. We refer the reader to [3,12] for more information about the $\overline{\partial}$-Neumann problem and related issues.

The commutators of the Bergman projection with multiplication operators can also be written in terms of the $\overline{\partial}$-Neumann operator. Let $f \in K^2_{(p,q)}(\Omega)$ and $\phi \in C^1(\overline{\Omega})$. Then the equality

$$[P_{p,q}, \phi]f = -\overline{\partial}^* N_{p,q+1}\overline{\partial}\phi \wedge f$$

follows easily from Kohn's formula. Furthermore, compactness of $N_{p,q+1}$ on $L^2_{(p,q+1)}(\Omega)$ implies that $[P_{p,q}, \phi]$ is compact on $L^2_{(p,q)}(\Omega)$ for all $\phi \in C(\overline{\Omega})$ (see [12, Propositions 4.1 and 4.2]). In the following theorem we show that the converse is true when $q \geq 1$.

Theorem 1. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, $0 \leq p \leq n$, and $1 \leq q \leq n-1$. Then the following are equivalent:*

i. *$N_{p,q+1}$ is compact on $L^2_{(p,q+1)}(\Omega)$,*
ii. *$\overline{\partial}^* N_{p,q+1}$ is compact on $K^2_{(p,q+1)}(\Omega)$,*
iii. *$[P_{p,q}, \bar{z}_j]$ is compact on $K^2_{(p,q)}(\Omega)$ for all $1 \leq j \leq n$,*
iv. *$[P_{p,q}, \bar{z}_j]$ is compact on $L^2_{(p,q)}(\Omega)$ for all $1 \leq j \leq n$,*
v. *$[P_{p,q}, \phi]$ is compact on $L^2_{(p,q)}(\Omega)$ for all $\phi \in C(\overline{\Omega})$,*
vi. *$[P_{p,q}, \phi]$ is compact on $K^2_{(p,q)}(\Omega)$ for all $\phi \in C(\overline{\Omega})$.*

The most important implications in Theorem 1 are iii. implies i. and iv. implies v. The rest of the implications are either known or easy: i. implies ii. is known [12, Proposition 4.2]; ii. implies iii. is easy; ii. implies iv. follows from [12, Proposition 4.1]; v. implies vi. and vi. implies iii. are obvious.

We would like to mention that Haslinger in [7, Theorem 3] proved equivalence of iii., iv., v., and vi. in Theorem 1 only in the case of $p = q = 0$.

One consequence of Theorem 1 is the following observation: To conclude that $N_{p,q}$ is compact it is enough to verify compactness of $\overline{\partial}^* N_{p,q}$ only, instead of verifying compactness of both $\overline{\partial}^* N_{p,q}$ and $\overline{\partial}^* N_{p,q+1}$ (see Proposition 1 in the next section).

Remark 1. We note that compactness of $N_{p,0}$ on the orthogonal complement of $A^2(\Omega)$ is equivalent to compactness of $N_{p,1}$ on $L^2_{(p,1)}(\Omega)$. This can be seen as follows: $N_{p,0} = \left(\overline{\partial}^* N_{p,1}\right)\left(\overline{\partial}^* N_{p,1}\right)^*$. This formula shows that compactness of $N_{p,0}$ implies compactness of $\overline{\partial}^* N_{p,1}$. Then Lemma 3 implies that $\overline{\partial}^* N_{p,2}$ is compact. Finally, Range's Formula (see [12, p. 77] and [10]) for $q = 1$

$$N_{p,1} = \left(\overline{\partial}^* N_{p,1}\right)^*\left(\overline{\partial}^* N_{p,1}\right) + \left(\overline{\partial}^* N_{p,2}\right)\left(\overline{\partial}^* N_{p,2}\right)^*$$

implies that $N_{p,1}$ is compact. On the other hand, compactness $N_{p,1}$ implies compactness of $\overline{\partial}^* N_{p,1}$. In turn, the formula $N_{p,0} = \left(\overline{\partial}^* N_{p,1}\right)\left(\overline{\partial}^* N_{p,1}\right)^*$ shows that, in this case, $N_{p,0}$ is compact on the orthogonal complement of $A^2(\Omega)$.