J. Math. Anal. Appl. 409 (2014) 399–408

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

# Multiple positive solutions for semi-linear elliptic systems with sign-changing weight

Haining Fan

*School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China*

ARTICLE INFO

*Article history:*
Received 11 April 2013
Available online 17 July 2013
Submitted by Manuel del Pino

*Keywords:*
Nehari manifold
Critical Sobolev exponent
Semi-linear elliptic system
Lusternik–Schnirelmann category
Multiple positive solutions

ABSTRACT

In this paper, we study the multiplicity results of positive solutions for a semi-linear elliptic system involving both concave–convex and critical growth terms. With the help of the Nehari manifold and the Lusternik–Schnirelmann category, we investigate how the coefficient $h(x)$ of the critical nonlinearity affects the number of positive solutions of that problem and get a relationship between the number of positive solutions and the topology of the global maximum set of $h$.



## 1. Introduction and the main result

This paper is concerned with the multiplicity of positive solutions to the following elliptic system:

$$(E_{f,g})\quad \begin{cases} -\Delta u = f(x)|u|^{q-2}u + \dfrac{\alpha}{\alpha+\beta}h(x)|u|^{\alpha-2}u|v|^{\beta}, & \text{in } \Omega, \\ -\Delta v = g(x)|v|^{q-2}v + \dfrac{\beta}{\alpha+\beta}h(x)|u|^{\alpha}|v|^{\beta-2}v, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $\alpha, \beta > 1$ satisfy $\alpha + \beta = 2^* = \frac{2N}{N-2}(N \geq 3)$ and $1 < q < 2$. Moreover, we assume that $f$, $g$ and $h$ satisfy the following conditions.

($H_1$) $f, g \in C(\overline{\Omega})$.
($H_2$) There exist a non-empty closed set $M = \{x \in \overline{\Omega};\, h(x) = \max_{x\in\overline{\Omega}} h(x) = 1\}$ and a positive number $\rho > 2$ when $N \geq 6$, $\rho > \frac{N-2}{2}$ when $3 \leq N \leq 5$ such that $h(z) - h(x) = O(|x - z|^{\rho})$ as $x \to z$ and uniformly in $z \in M$.
($H_3$) $f(x), g(x) > 0$ for $x \in M$.

**Remark 1.1.** Let $M_r = \{x \in \mathbb{R}^N;\, dist(x, M) < r\}$ for $r > 0$. Then by ($H_1$)–($H_3$), there exist $C_0, r_0 > 0$ such that

$$f(x), g(x), h(x) > 0 \quad \text{for all } x \in M_{r_0} \subset \Omega$$

and

$$h(z) - h(x) \leq C_0|x - z|^{\rho} \quad \text{for all } x \in B_{r_0}(z)$$

uniformly in $z \in M$, where $B_{r_0}(z) = \{x \in \mathbb{R}^N;\, |x - z| < r_0\}$.

*E-mail address:* fanhaining888@163.com.

0022-247X/$ – see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.jmaa.2013.07.014

For the systems of semi-linear elliptic equations with concave–convex nonlinearities, various studies concerning the solution structures have been presented (for example [10,1,15,4,3,8,5]). In particular, for $f \equiv \lambda$, $g \equiv \mu$, Hsu [10] proved that $(E_{f,g})$ permits at least two positive solutions when the pair of parameters $(\lambda, \mu)$ belongs to a certain subset of $\mathbb{R}^2$. Similar results were obtained by Adriouch and El Hamidi [1]. Further studies involving sign-changing weight functions were taken by Wu [15] and Chen and Wu [4] for example, where the two positive solutions were obtained for the subcritical case $2 < \alpha + \beta < 2^*$ in [15] while these for the critical case $\alpha + \beta = 2^*$ were obtained in [4]. The tool of them is the decomposition of the Nehari manifold.

For $2 < q < 2^*$, if $N > 4$, $0 \in \Omega$, $f$, $g$ and $h$ satisfy the following conditions.

($A_1$) $f$, $g$ and $h$ are positive continuous functions in $\overline{\Omega}$.

($A_2$) There exist $k$ points $a^1, a^2, \ldots, a^k$ in $\Omega$ such that

$$h(a^i) = \max_{x\in\overline{\Omega}} h(x) = 1 \quad \text{for } 1 \le i \le k,$$

and for some $\rho > N$, $h(x) - h(a^i) = O(|x - a^i|^\rho)$ as $x \to a^i$ and uniformly in $i$.

($A_3$) Choose $\rho_0 > 0$ such that

$$\overline{B_{\rho_0}(a^i)} \bigcap \overline{B_{\rho_0}(a^j)} = \emptyset \quad \text{for } i \neq j \text{ and } 1 \le i, j \le k,$$

and $\bigcup_{i=1}^k \overline{B_{\rho_0}(a^i)} \subset \Omega$, where $\overline{B_{\rho_0}(a^i)} = \{x \in \mathbb{R}^N;\ |x - z| \le \rho_0\}$.

Lin [12] recently proved that $(E_{f,g})$ admits at least $k$ positive solutions when $f$ and $g$ are small enough. A similar result was obtained in Li and Yang [11].

Motivated by [12,11], we aim to investigate how the coefficient $h(x)$ of the critical nonlinearity affects the number of positive solutions of $(E_{f,g})$ when $1 < q < 2$ in this work. We try to consider the relationship between the number of positive solutions and the topology of the global maximum set of $h$ by the idea of category. Furthermore, by borrowing some techniques from [10,1,15,4,3,8,5], we will study $(E_{f,g})$ under the conditions ($H_1$)–($H_3$), i.e., we do not need to assume $f$, $g$, $h$ are positive solutions and $0 \in \Omega$ as [12,11]. The main result of this paper is as follows.

**Theorem 1.1.** *Assume* ($H_1$)–($H_3$) *hold. Then for each* $\delta < r_0$*, there exists* $\Lambda_\delta > 0$ *such that if* $\|f_+\|_{L^{q^*}} + \|g_+\|_{L^{q^*}} < \Lambda_\delta$, $(E_{f,g})$ *has at least* $\operatorname{cat}_{M_\delta}(M) + 1$ *distinct positive solutions, where* $f_+ = \max\{f, 0\}$, $g_+ = \max\{g, 0\}$, $q^* = \frac{2^*}{2^*-q}$ *and cat means the Lusternik–Schnirelmann category (see* [13]*).*

**Remark 1.2.** Suppose ($A_1$)–($A_3$) hold. By Theorem 1.1, we obtain that $(E_{f,g})$ has at least $k+1$ positive solutions when $\|f\|_{L^{q^*}}$ and $\|g\|_{L^{q^*}}$ are small enough.

This paper is organized as follows. In Section 2, we give some notations and preliminary results. In Section 3, we discuss some concentration behavior. In Section 4, we prove Theorem 1.1.

## 2. Notations and preliminaries

We propose to study $(E_{f,g})$ in the framework of the Sobolev space $H = H_0^1(\Omega) \times H_0^1(\Omega)$ using the standard norm

$$\|(u, v)\|_H = \left(\int_\Omega |\nabla u|^2 + |\nabla v|^2 dx\right)^{\frac{1}{2}}.$$

Denote

$$S_{\alpha,\beta} := \inf_{(u,v)\in H\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 + |\nabla v|^2 dx}{\left(\int_\Omega |u|^\alpha |v|^\beta dx\right)^{\frac{2}{\alpha+\beta}}}.$$

Working as in the proof of [2, Theorem 5], we deduce that

$$S_{\alpha,\beta} = \left(\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right) S,$$

where $S$ is the best Sobolev constant, that is

$$S := \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^{2^*} dx\right)^{\frac{2}{2^*}}}.$$

It is well known that $S$ is independent of $\Omega$, and for each $\varepsilon > 0$,

$$v_\varepsilon(x) = \frac{[N(N-2)\varepsilon^2]^{(N-2)/4}}{(\varepsilon^2 + |x|^2)^{(N-2)/2}} \tag{2.1}$$