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Relaxation in nonconvex optimal control problems described by fractional differential equations☆

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ABSTRACT

We consider the minimization problem of an integral functional with integrand that is not convex in the control on solutions of a control system described by fractional differential equation with mixed nonconvex constraints on the control. A relaxation problem is treated along with the original problem. It is proved that, under general assumptions, the relaxation problem has an optimal solution, and that for each optimal solution there is a minimizing sequence of the original problem that converges to the optimal solution with respect to the trajectory, the control, and the functional in appropriate topologies simultaneously.

1. Introduction

In this article, we are interested in control systems described by fractional differential equations of the type

$$\begin{cases} {}^C D_t^\alpha x(t) = Ax(t) + B(t)u(t), & t \in J = [0, b], \text{ with } 0 < \alpha < 1, \\ x(0) = x_0, \end{cases} \tag{1}$$

with mixed nonconvex constraints on the control,

$$u(t) \in U(t, x(t)) \quad \text{a.e. on } J, \tag{2}$$

where ${}^C D_t^\alpha$ is the Caputo fractional derivative of order α, $b > 0$ is a finite real number, A is the infinitesimal generator of a strongly continuous semigroup $\{T(t), t \geq 0\}$ in a Banach space X, $B : J \to \mathcal{L}(Y, X)$ ($\mathcal{L}(Y, X)$ is the space of continuous linear operators from Y into X), and $U : J \times X \to 2^Y \setminus \{\emptyset\}$ is a multivalued map with closed values that is not necessarily convex. The space Y is a separable reflexive Banach space modeling the control space.

Let $\bar{\mathbb{R}} = (-\infty, +\infty]$. For a numerical function $g : J \times X \times Y \to \bar{\mathbb{R}}$, we consider problem (P):

$$I(x, u) = \int_J g(t, x(t), u(t))dt \to \inf \tag{P}$$

on solutions of the control system (1) with constraint (2).

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Let $g_U : J \times X \times Y \to \bar{\mathbb{R}}$ be the function defined by

$$g_U(t,x,u) = \begin{cases} g(t,x,u), & u \in U(t,x), \\ +\infty, & u \notin U(t,x), \end{cases}$$

and let $g^{**}(t,x,u)$ be the bipolar of the function $u \to g_U(t,x,u)$ [6]. Along with problem (P), we also consider the relaxation problem (RP):

$$I^{**}(x,u) = \int_J g^{**}(t,x(t),u(t))dt \to \inf \tag{RP}$$

on the solutions of control system (1) with the convexified constraints

$$u(t) \in \overline{\text{co}}U(t,x(t)) \quad \text{a.e. on } J \tag{3}$$

on the control. Here, $\overline{\text{co}}$ stands for the closed convex hull of a set.

The aim of this paper is to establish an interrelation between the solutions of problem (P) and problem (RP). Under certain assumptions, it is proved that (RP) has a solution and that for any solution of (RP) there is a minimizing sequence for (P) converging in the appropriate topologies to the solution of (RP). The convergence takes place simultaneously with respect to the trajectory, the control, and the functional. This property is usually called relaxation [6]. The main result obtained in our work is the following theorem.

Theorem 1.1. *Problem (RP) has a solution, and*

$$\min_{(x,u)\in\mathcal{R}_{\overline{\text{co}}U}} I^{**}(x,u) = \inf_{(x,u)\in\mathcal{R}_U} I(x,u). \tag{4}$$

For any solution (x_*,u_*) of problem (RP) there exists a minimizing sequence $(x_n,u_n) \in \mathcal{R}_U$, $n \geq 1$ for problem (P) which converges to (x_*,u_*) in the spaces $C(J,X) \times \omega\text{-}L^{\frac{1}{\beta}}(J,Y)$ and $C(J,X) \times L_\omega^{\frac{1}{\beta}}(J,Y)$, and the following formula holds:

$$\lim_{n\to\infty} \sup_{0\le t_1\le t_2\le b} \left| \int_{t_1}^{t_2} \left(g^{**}(s,x_*(s),u_*(s)) - g(s,x_n(s),u_n(s))\right) ds \right| = 0. \tag{5}$$

Conversely, if (x_n,u_n), $n \geq 1$ is a minimizing sequence for problem (P), then there is a subsequence (x_{n_k},u_{n_k}), $k \geq 1$ of the sequence (x_n,u_n), $n \geq 1$, and a solution (x_*,u_*) of problem (RP) such that the subsequence (x_{n_k},u_{n_k}), $k \geq 1$, converges to (x_*,u_*) in $C(J,X) \times \omega\text{-}L^{\frac{1}{\beta}}(J,Y)$ and relation (5) holds for this subsequence (x_{n_k},u_{n_k}), $k \geq 1$.

Here, $\mathcal{R}_U$ and $\mathcal{R}_{\overline{\text{co}}U}$ denote the sets of all solutions of control system (1), (2) and control system (1), (3), respectively.

The results obtained in this paper are an analogue of the classical Bogolyubov theorem [3,4,25] in the calculus of variations with constraints being the sets of solutions of control systems (1), (2) and (1), (3), which also allow us to justify, while performing numerical calculations, the passage from a nonconvex optimal control problem to the convexified optimal control problem and the approximation of the latter by a sequence of smooth and convex optimal control problems for which the optimality conditions are known and methods of their numerical resolution are well developed.

Fractional differential equations have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc.; see [7–9,14,17,15], for example. There has been significant development in fractional differential equations in recent years; see the monographs of Kilbas et al. [12] and Miller et al. [18], and the references therein. As for the study of fractional semilinear differential equations, we can refer to [31,32,29] for the existence results. Approximate controllability was considered in [16,13,22], and [30] is concerned with optimal control theory.

For the control problem of semilinear evolution inclusions, see, for example, [20,19]. For optimal control problems similar to ours of the subdifferential type, see, for example, [21,26,24].

In the next section, we will introduce some useful preliminaries and give assumptions on the data. In Section 3, some auxiliary results needed in the proof of our main results are presented. Section 4 deals with the existence results of the control systems. The main results are proved in Section 5.

2. Preliminaries and assumptions

Let $J = [0,b]$ be the closed interval of the real line with Lebesgue measure μ and σ-algebra Σ of μ measurable sets. The norm of the space X (or Y) will be denoted by $\|\cdot\|_X$ (or $\|\cdot\|_Y$). We denote by $C(J,X)$ the space of all continuous functions from J into X with the supnorm given by $\|x\|_C = \sup_{t\in J}\|x(t)\|_X$ for $x \in C(J,X)$. For a Banach space X, the symbol ω-X stands for X equipped with the weak $\sigma(X,X^*)$ topology. The same notation will be used for subsets of X. In all other cases we assume that X and its subsets are equipped with the strong (normed) topology.