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On limit directions of Julia sets of entire solutions of linear differential equations☆

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ABSTRACT

This paper is devoted to studying the limit directions of Julia sets of solutions of $f^{(n)} + A_{n-1}(z)f^{(n-1)} + \cdots + A_0(z)f = 0$, where $n\ (\geq 2)$ is an integer and $A_j(z)\ (j = 0, 1, \ldots, n-1)$ are entire functions of finite lower order. With some additional conditions on coefficients, we know that every non-trivial solution $f(z)$ of such equations has infinite lower order, and prove that the limit directions of Julia sets of $f(z)$ must have a definite range of measure.

1. Introduction and main results

Nevanlinna theory is an important tool in this paper; its usual notations and basic results come mainly from [7,8,11]. Let f be a meromorphic function in the whole complex plane. We use $\lambda(f)$ and $\mu(f)$ to denote the order and the lower order of f, respectively, which are defined as [21, Definition 1.6]

$$\lambda(f) = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}, \qquad \mu(f) = \liminf_{r\to\infty} \frac{\log^+ T(r,f)}{\log r},$$

and denote the deficiency of f at the point a by (see [11, p. 46])

$$\delta(a,f) = \liminf_{r\to\infty} \frac{m\left(r, \frac{1}{f-a}\right)}{T(r,f)} = 1 - \limsup_{r\to\infty} \frac{N\left(r, \frac{1}{f-a}\right)}{T(r,f)}.$$

We define f^n, $n \in \mathbb{N}$ as the nth iterate of f; that is, $f^1 = f, \ldots, f^n = f \circ (f^{n-1})$. The Fatou set $F(f)$ of f is the subset of $\mathbb{C}$ where $\{f^n(z)\}_{n=1}^{\infty}$ forms a normal family, and its complement $J(f) = \mathbb{C} \setminus F(f)$ is called the Julia set of f. It is well known that $F(f)$ is open and completely invariant under f, and $J(f)$ is closed and non-empty. For an introduction to the dynamics of meromorphic functions, we refer the reader to Bergweiler's paper [6] and Zheng's book [22].

Assuming that $0 < \alpha < \beta < 2\pi$, we denote $\Omega(\alpha, \beta) = \{z \in \mathbb{C} \mid \arg z \in (\alpha, \beta)\}$. Given $\theta \in [0, 2\pi)$, if $\Omega(\theta - \varepsilon, \theta + \varepsilon) \cap J(f)$ is unbounded for every $\varepsilon > 0$, we say that the radial $\arg z = \theta$ is a limit direction of $J(f)$. Define

$$\Delta(f) = \{\theta \in [0, 2\pi) : \text{the radial } \arg z = \theta \text{ is a limit direction of } J(f)\}.$$

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Clearly, $\Delta(f)$ is closed, so it is measurable, and we use $mes\Delta(f)$ to denote its linear measure. Baker [2] first observed that, for a transcendental entire function f, $J(f)$ cannot lie in finitely many rays emanating from the origin. For the case that $f(z)$ is a transcendental entire function of finite lower order, Qiao [15] proved that $mes\Delta(f) = 2\pi$ if $\mu(f) < 1/2$ and $mes\Delta(f) \geq \pi/\mu(f)$ if $\mu(f) \geq 1/2$. Moreover, Qiao, in [16], obtained some further results for transcendental entire functions. Later, some observations on limit directions of the Julia sets were made to transcendental meromorphic functions with finite lower order; see [25,17]. Naturally, a question can be posed here: what can we say about the limit directions of entire functions with infinite lower order?

Baker [2] constructed an entire function, for every $M > 0$, of infinite lower order satisfying $J(f) \subset \{z \in \mathbb{C} : |Imz| < M, Rez > 0\}$. Thus $\Delta(f) = \{0\}$. Recently, Huang and Wang [9] investigated the limit directions of the product of linearly independent solutions of equations

$$f^{(n)} + A(z)f = 0, \tag{1}$$

where $A(z)$ is a transcendental entire function with finite order. Their result is stated as follows.

Theorem A. *Let $\{f_1, f_2, \ldots, f_n\}$ be a solution base of Eq. (1), and denote $E = f_1 f_2 \cdots f_n$. Then*

$$mes\Delta(E) \geq \min\{2\pi, \pi/\lambda(A)\}.$$

Even though every non-trivial solution of Eq. (1) is an entire function of infinite lower order, $E(z)$ may have finite lower order. For example, the equation $f'' - \frac{1}{4}(e^{2z} + 1)f = 0$ admits two linearly independent solutions,

$$f_1 = \exp\left\{-\frac{1}{2}(z + e^z)\right\}, \qquad f_2 = \exp\left\{-\frac{1}{2}(z - e^z)\right\},$$

so $\mu(E) = \mu(e^{-z}) = 1$. In some cases, $E(z)$ can be an entire function of infinite lower order. For example, for the equation $f'' - (e^{2z} + e^z)f = 0$, see [4, p. 394], we have

$$4(e^{2z} + e^z) = \left(\frac{c}{E}\right)^2 - \left(\frac{E'}{E}\right)^2 + 2\frac{E''}{E}, \tag{2}$$

where c is a constant(the Wronskian of f_1 and f_2). By using Wiman–Valiron theory [10,12,11] and the behavior of $e^{p(z)}$ [13, p. 254], where $p(z)$ is a non-constant polynomial, we can conclude from Eq. (2) that $E(z)$ must have infinite lower order.

In this paper, by using the spread relation of meromorphic functions, we will study the limit directions of solutions of linear differential equations directly.

Theorem 1.1. *Let $A_i(z)$ $(i = 0, 1, \ldots, n-1)$ be entire functions of finite lower order such that A_0 is transcendental and $m(r, A_i) = o(m(r, A_0))$ $(i = 1, 2, \ldots, n-1)$ as $r \to \infty$. Then every non-trivial solution f of the equation*

$$f^{(n)} + A_{n-1}f^{(n-1)} + \cdots + A_0 f = 0 \tag{3}$$

satisfies $mes\Delta(f) \geq \min\{2\pi, \pi/\mu(A_0)\}$.

Corollary 1.1. *Suppose that $A_i(z)$ $(i = 0, 1, \ldots, n-1)$ are entire functions satisfying $\lambda(A_j) < \mu(A_0)$ $(j = 1, 2, \ldots, n-1)$ and $\mu(A_0) < +\infty$ Then, for every non-trivial solution f of Eq. (3), we have $mes\Delta(f) \geq \min\{2\pi, \pi/\mu(A_0)\}$.*

Since $A_0(z)$ is entire and transcendental,

$$\lim_{r\to\infty} \frac{m(r, A_0)}{\log r} = \infty.$$

Applying the lemma of logarithmic derivatives to Eq. (3) yields

$$\begin{aligned} m(r, A_0) &\leq \sum_{i=1}^{n} m\left(r, \frac{f^{(i)}}{f}\right) + \sum_{i=1}^{n-1} m(r, A_i) + O(1) \\ &= O(\log T(r, f) + \log r) + o(m(r, A_0)), \end{aligned}$$

outside of a possible exceptional set E of finite linear measure. Therefore, all non-trivial solutions of Eq. (3) are entire functions with infinite lower order.

It is difficult to prove Theorem 1.1 by using similar reasoning to that in the proof of Theorem A, since that method depends strongly on the fact that the Wronskian determinant of the solution base of Eq. (1) is a constant, while this is not true for Eq. (3). Instead, we will apply the spread relation and Pólya peaks of meromorphic functions in this paper.

Moreover, if $A_0(z)$ is of finite order, then we can obtain a more detailed result, which is stated as follows.

Theorem 1.2. *Suppose that $A_i(z)(i = 0, 1, \ldots, n-1)$ are entire functions of finite order and that $\lambda(A_i) < \lambda(A_0)$ $(i = 1, 2, \ldots, n-1)$. Then there exists a closed interval $I \in [0, 2\pi)$ such that, for every non-trivial solution f of Eq. (3), we have $I \subset \Delta(f)$ and $mesI \geq \min\{2\pi, \pi/\lambda(A_0)\}$.*