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A continuous method for nonlocal functional differential equations with delayed or advanced arguments

X.Y. Li [a], B.Y. Wu [b,*]

[a] Department of Mathematics, Changshu Institute of Technology, Suzhou, Jiangsu 215500, China
[b] Department of Mathematics, Harbin Institute of Technology, Harbin, Heilongjiang 150001, China

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ABSTRACT

In the previous works, the authors presented the reproducing kernel method (RKM) for solving various differential equations. However, to the best of our knowledge, there exist no results for functional differential equations. The aim of this paper is to extend the application of reproducing kernel theory to nonlocal functional differential equations with delayed or advanced arguments, and give the error estimation for the present method. Some numerical examples are provided to show the validity of the present method.

1. Introduction

Reproducing kernel theory has important application in numerical analysis, differential equations, probability and statistics, learning theory and so on. Recently, reproducing kernel methods for solving a variety of differential equations were presented by Cui and Geng [4–9], Lin and Zhou [12,13], Yao, Chen and Jiang [3,26], Wang, Li and Wu [11,22], Mohammadi and Mokhtari [15], Akram and Ur Rehman [1], Arqub, Al-Smadi, Momani [2], Özen and Oruçoğlu [17], Wang, Han and Yamamoto [23].

In this paper, we consider the following functional differential equations with linear nonlocal conditions:

$$\begin{cases} u'(x) + a(x)u(x) + b(x)u(\tau(x)) = f(x), \quad x \in I = [0, 1], \\ B(u(c), u) = 0, \end{cases} \tag{1.1}$$

where $c \in I$, $a(x)$, $b(x) \in C[0, 1]$, $\tau(x) \in C^1[0, 1]$, and f is given such that (1.1) satisfies the existence and uniqueness of the solutions.

Notice that function $B(u(c), u) = 0$ includes several types of boundary conditions: initial conditions, final conditions, periodic conditions, or more general functional conditions, as

$$B(u(c), u) = \lambda u(c) - \int_0^1 u(s)ds, \quad \lambda \in R.$$

Functional differential equations arise in a variety of applications, such as number theory, electrodynamics, astrophysics, nonlinear dynamical systems, probability theory on algebraic structure, quantum mechanics and cell growth. Therefore, the problems have attracted a great deal of attention. Liu and Li [14] studied the analytical and numerical solutions of the multi-pantograph equations. Sezer [20,21] gave the series solutions of multi-pantograph equations with variable coefficients.

* Corresponding author.
E-mail addresses: lixiuying1112@126.com (X.Y. Li), mathwby@sina.com (B.Y. Wu).

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Wang, Qin and Li [25] proposed one-leg h-methods for nonlinear neutral differential equations with proportional delay. In [10,16,19], the authors discussed the series solutions of some functional differential equations. Wang, Shen and Luo [24] obtained the existence of solutions of second-order multi-point functional differential equations by using the method of upper and lower solutions. Rodríguez-López [18] studied the existence of a solution to a nonlocal boundary value problem for a class of second-order functional differential equations with piecewise constant arguments.

However, numerical solutions of nonlocal functional differential equations are seldom discussed. The aim of this paper is to fill this gap.

The rest of the paper is organized as follows. In the next section, the reproducing kernel method for solving (1.1) is introduced. The error estimation is presented in Section 3. Numerical examples are provided in Section 4. Section 5 ends this paper with a brief conclusion.

2. Reproducing kernel method for (1.1)

In this section, we extend the application of reproducing kernel theory to nonlocal functional differential equation (1.1).

To solve (1.1), first, we construct reproducing kernel spaces $W^m[0, 1]$, $(m \geq 2)$ in which every function satisfies the boundary condition of (1.1).

Definition 2.1. $W_0^m[0, 1] = \{u(x) \mid u^{(m-1)}(x)$ is an absolutely continuous real value function, $u^{(m)}(x) \in L^2[0, 1]\}$. The inner product and norm in $W_0^m[0, 1]$ are given respectively by

$$(u, v)_m = \sum_{i=0}^{m-1} u^{(i)}(0)v^{(i)}(0) + \int_0^1 u^{(m)}(x)v^{(m)}(x)dx$$

and

$$\|u\|_m = \sqrt{(u, u)_m}, \quad u, v \in W_0^m[0, 1].$$

By [6,8], $W_0^m[0, 1]$ is a reproducing kernel space and its reproducing kernel $k_0(x, y)$ can be obtained.

Next, we construct reproducing kernel space $W^m[0, 1]$ in which every function satisfies $B(u(c), u) = 0$.

Definition 2.2. $W^m[0, 1] = \{u(x) \mid u(x) \in W_0^m[0, 1], B(u(c), u) = 0\}$.

Clearly, $W^m[0, X]$ is a closed subspace of $W_0^m[0, X]$ and therefore it is also a reproducing kernel space.

Put $Pu(x) = B(u(c), u)$.

Theorem 2.1. *If $P_xP_yk(x, y) \neq 0$, then the reproducing kernel $K(x, y)$ of $W^m[0, 1]$ is given by*

$$K(x, y) = k_0(x, y) - \frac{P_xk_0(x, y)P_yk_0(x, y)}{P_xP_yk_0(x, y),} \tag{2.1}$$

where the subscript x by the operator P indicates that the operator P applies to the function of x.

Proof. It is easy to see that $PK(x, y) = 0$, and therefore $K(x, y) \in W^m[0, 1]$.

For all $u(y) \in W^m[0, 1]$, obviously, $P_yu(y) = 0$, it follows that

$$(u(y), K(x, y))_m = (u(y), k_0(x, y))_m = u(x).$$

That is, $K(x, y)$ is of "reproducing property". Thus, $K(x, y)$ is the reproducing kernel of $W^m[0, 1]$ and the proof is complete. □

In [6], Cui and Lin defined reproducing kernel space $W^1[0, 1]$ and gave its reproducing kernel

$$\bar{k}(x, y) = \begin{cases} 1 + y, & y \leq x, \\ 1 + x, & y > x. \end{cases}$$

Put

$$Lu(x) = u'(x) + a(x)u(x) + b(x)u(\tau(x)).$$

Theorem 2.2. *$L : W^m[0, 1] \to W^1[0, 1]$ is a bounded linear operator.*

Proof. It is easy to see that

$$\begin{aligned} |u(x)| &= |(u(\cdot),\ K(x, \cdot))_m| \\ &\leq \|u(\cdot)\|_m\ \|K(x, \cdot)\|_m. \end{aligned} \tag{2.2}$$