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Inequalities for the generalized trigonometric and hyperbolic functions

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ABSTRACT

The generalized trigonometric functions occur as an eigenfunction of the Dirichlet problem for the one-dimensional p-Laplacian. The generalized hyperbolic functions are defined similarly. Some classical inequalities for trigonometric and hyperbolic functions, such as Mitrinović–Adamović's inequality, Lazarević's inequality, Huygens-type inequalities, Wilker-type inequalities, and Cusa–Huygens-type inequalities, are generalized to the case of generalized functions.

1. Introduction

It is well known from basic calculus that

$$\arcsin(x) = \int_0^x \frac{1}{(1-t^2)^{1/2}}\,dt, \quad 0 \le x \le 1,$$

and

$$\frac{\pi}{2} = \arcsin(1) = \int_0^1 \frac{1}{(1-t^2)^{1/2}}\,dt.$$

We can define the function sin on $[0, \pi/2]$ as the inverse of arcsin and extend it on $(-\infty, \infty)$.

Let $1 < p < \infty$. We can generalize the above functions as follows:

$$\operatorname{arcsin}_p(x) \equiv \int_0^x \frac{1}{(1-t^p)^{1/p}}\,dt, \quad 0 \le x \le 1,$$

and

$$\frac{\pi_p}{2} = \operatorname{arcsin}_p(1) \equiv \int_0^1 \frac{1}{(1-t^p)^{1/p}}\,dt.$$

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The inverse of $\arcsin_p$ on $[0, \pi_p/2]$ is called the *generalized sine function* and denoted by $\sin_p$. By standard extension procedures as the sine function we get a differentiable function on the whole of $(-\infty, \infty)$ which coincides with sin when $p = 2$. It is easy to see that the function $\sin_p$ is strictly increasing and concave on $[0, \pi_p/2]$. In the same way we can define the generalized cosine function, the generalized tangent function, and their inverses, and also the corresponding hyperbolic functions.

The generalized sine function $\sin_p$ occurs as an eigenfunction of the Dirichlet problem for the one-dimensional p-Laplacian. There are several different definitions for these generalized trigonometric and hyperbolic functions [8,10–12]. Recently, these functions have been studied very extensively (see [4–6,8,10–13]). In particular, the reader is referred to [10–13]. These generalized functions are similar to the classical functions in various aspects. Some of these functions can be expressed in terms of the Gaussian hypergeometric series (see [3,4]).

In this paper we will generalize some classical inequalities for trigonometric and hyperbolic functions, such as *Mitrinović–Adamović's inequality* (Theorem 3.6), *Lazarević's inequality* (Theorem 3.8), *Huygens-type inequalities* (Theorems 3.13 *and* 3.16), *Wilker-type inequality* (Corollary 3.19), and *Cusa–Huygens-type inequalities* (Theorems 3.22 and 3.24) to the case of generalized functions. For the classical cases, these inequalities have been extended and sharpened extensively (see the very recent survey [2]).

2. Definitions and formulas

In this section we define the generalized cosine function, the generalized tangent function, and their inverses, and also the corresponding hyperbolic functions.

The *generalized cosine function* $\cos_p$ is defined as

$$\cos_p(x) \equiv \frac{d}{dx}\sin_p(x).$$

It is clear from the definitions that

$$\cos_p(x) = (1 - \sin_p(x)^p)^{1/p}, \quad x \in [0, \pi_p/2],$$

and

$$|\sin_p(x)|^p + |\cos_p(x)|^p = 1, \quad x \in \mathbb{R}. \tag{2.1}$$

It is easy to see that

$$\frac{d}{dx}\cos_p(x) = -\cos_p(x)^{2-p}\sin_p(x)^{p-1}, \quad x \in [0, \pi_p/2].$$

The *generalized tangent function* is defined as in the classical case:

$$\tan_p(x) \equiv \frac{\sin_p(x)}{\cos_p(x)}, \quad x \in \mathbb{R} \setminus \left\{k\pi_p + \frac{\pi_p}{2} : k \in \mathbb{Z}\right\}.$$

It follows from (2.1) that

$$\frac{d}{dx}\tan_p(x) = 1 + |\tan_p(x)|^p, \quad x \in (-\pi_p/2, \pi_p/2).$$

Similarly, the *generalized inverse hyperbolic sine function*

$$\operatorname{arcsinh}_p(x) \equiv \begin{cases} \displaystyle\int_0^x \frac{1}{(1+t^p)^{1/p}}\,dt, & x \in [0, \infty), \\ -\operatorname{arcsinh}_p(-x), & x \in (-\infty, 0) \end{cases}$$

generalizes the classical inverse hyperbolic sine function. The inverse of $\operatorname{arcsinh}_p$ is called the *generalized hyperbolic sine function* and denoted by $\sinh_p$. The *generalized hyperbolic cosine function* is defined as

$$\cosh_p(x) \equiv \frac{d}{dx}\sinh_p(x).$$

The definitions show that

$$\cosh_p(x)^p - |\sinh_p(x)|^p = 1, \quad x \in \mathbb{R},$$

and

$$\frac{d}{dx}\cosh_p(x) = \cosh_p(x)^{2-p}\sinh_p(x)^{p-1}, \quad x \geq 0.$$

The *generalized hyperbolic tangent function* is defined as

$$\tanh_p(x) \equiv \frac{\sinh_p(x)}{\cosh_p(x)},$$