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On the perturbation determinants of a singular dissipative boundary value problem with finite transmission conditions

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ABSTRACT

In this paper a singular dissipative boundary value problem with finite transmission conditions is investigated. Using Livšic's theorem, it is proved that the system of all eigen and associated functions of this problem is complete in the Hilbert space.

1. Introduction

Let us consider the differential expression

$$\eta(y)=\frac{1}{w(x)}\left[-(p(x)y')'+q(x)y\right],\quad x\in\Lambda:=\bigcup_{k=1}^{n+1}\Lambda_k,$$

where $\Lambda_k=(c_{k-1},c_k)$ and $-\infty<c_0<c_1<\cdots<c_{n+1}\leq\infty$. In this paper the following properties are assumed to be satisfied:

(i) the points $c_0, c_1, \ldots, c_n$ are regular and c_{n+1} is singular for the differential expression η,
(ii) p, q and w are real-valued, Lebesgue measurable functions on Λ,
(iii) $p^{-1}, q, w\in L^1_{\mathrm{loc}}(\Lambda_k),\ k=1,2,\ldots,n+1$, and
(iv) $w(x)>0$ for almost all x on Λ.

Let $L^2_w(\Lambda)$ denote the Hilbert space consisting of all complex valued functions y such that $\int_\Lambda w(x)\,|y(x)|^2\,dx<\infty$ with the inner product

$$(y,\chi)=\int_\Lambda w(x)y(x)\overline{\chi}(x)dx.$$

Let D denote the set of all functions $y\in L^2_w(\Lambda)$ such that y, py' are locally absolutely continuous functions on all $\Lambda_k,\ k=1,2,\ldots,n+1$, and $\eta(y)\in L^2_w(\Lambda)$. For arbitrary $y,\chi\in D$, Green's formula is

$$\int_\Lambda w(x)\eta(y)\overline{\chi}(x)dx-\int_\Lambda w(x)y(x)\overline{\eta(\chi)}dx=\sum_{k=1}^{n+1}[y,\chi]_{c_{k-1}+}^{c_k-},$$

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where $[y,\chi]_{c_{k-1}+}^{c_k-}=[y,\chi]_{c_k-}-[y,\chi]_{c_{k-1}+}$, $[y,\chi]_x:=y(x)\overline{\chi^{[1]}}(x)-y^{[1]}(x)\overline{\chi}(x)$ and $y^{[1]}$ denotes py'. This equation implies that at singular point c_{n+1} for all $y,\chi\in D$, the limit $[y,\chi]_{c_{n+1}}:=[y,\chi]_{c_{n+1}-}=\lim_{x\to c_{n+1}-}[y,\chi]_x$ exists and is finite.

In this paper it is assumed that the functions p,q and w satisfy Weyl's limit-circle case conditions at singular point c_{n+1}. Weyl's theory is well known and there are several sufficient conditions in which Weyl's limit-circle case holds for a differential expression [2,5,8,12,13,16,17].

Now let us consider the solutions $\varphi(x,\lambda)=\{\varphi_1(x,\lambda),\varphi_2(x,\lambda),\dots,\varphi_{n+1}(x,\lambda)\}$ and $\psi(x,\lambda)=\{\psi_1(x,\lambda),\psi_2(x,\lambda),\dots,\psi_{n+1}(x,\lambda)\}$, where $\varphi_k(x,\lambda)$ and $\psi_k(x,\lambda)$ are the parts of the functions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$, respectively, defined on the interval Λ_k $(k=1,2,\dots,n+1)$, of the equation

$$-(p(x)y')'+q(x)y=\lambda w(x)y,\quad x\in\Lambda, \tag{1.1}$$

where λ is some complex parameter, satisfying the conditions [1,4,10,11,14],

$$\begin{cases}\varphi_1(c_0,\lambda)=\cos\alpha, & \varphi_1^{[1]}(c_0,\lambda)=\sin\alpha,\\ \psi_1(c_0,\lambda)=-\sin\alpha, & \psi_1^{[1]}(c_0,\lambda)=\cos\alpha,\end{cases}$$

and

$$\begin{cases}\varphi_{m+1}(c_m+,\lambda)=\dfrac{1}{\kappa_m}\varphi_m(c_m-,\lambda), & \varphi_{m+1}^{[1]}(c_m+,\lambda)=\dfrac{1}{\kappa_m'}\varphi_m^{[1]}(c_m-,\lambda),\\ \psi_{m+1}(c_m+,\lambda)=\dfrac{1}{\kappa_m}\psi_m(c_m-,\lambda), & \psi_{m+1}^{[1]}(c_m+,\lambda)=\dfrac{1}{\kappa_m'}\psi_m^{[1]}(c_m-,\lambda),\end{cases}$$

where α,κ_m and κ_m' are some real numbers with $\kappa_m\kappa_m'>0$ and $m=1,2,\dots,n$. Since Weyl's limit-circle case holds at singular point c_{n+1} for η, the solutions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ $(x\in\Lambda)$ belong to $L_w^2(\Lambda)$.

Let $z(x)=\{z_1(x),z_2(x),\dots,z_{n+1}(x)\}$ and $u(x)=\{u_1(x),u_2(x),\dots,u_{n+1}(x)\}$ be the solutions of $\eta(y)=0$ $(x\in\Lambda)$ satisfying the conditions

$$\begin{cases}z_1(c_0)=\cos\alpha, & z_1^{[1]}(c_0)=\sin\alpha,\\ u_1(c_0)=-\sin\alpha, & u_1^{[1]}(c_0)=\cos\alpha,\end{cases}$$

and

$$\begin{cases}z_{m+1}(c_m+)=\dfrac{1}{\kappa_m}z_m(c_m-), & z_{m+1}^{[1]}(c_m+)=\dfrac{1}{\kappa_m'}z_m^{[1]}(c_m-),\\ u_{m+1}(c_m+)=\dfrac{1}{\kappa_m}u_m(c_m-), & u_{m+1}^{[1]}(c_m+)=\dfrac{1}{\kappa_m'}u_m^{[1]}(c_m-),\end{cases}$$

where α,κ_m and κ_m' are some real numbers with $\kappa_m\kappa_m'>0$ and $m=1,2,\dots,n$. It is clear that $z(x)=\varphi(x,0)$ $(x\in\Lambda)$ and $u(x)=\psi(x,0)$ $(x\in\Lambda)$. Hence $z(x)$ and $u(x)$ belong to $L_w^2(\Lambda)$. Further they belong to D. This implies that for each $y\in D$, at singular point c_{n+1}, the values $[y,z]_{c_{n+1}}$ and $[y,u]_{c_{n+1}}$ exist and are finite.

It should be noted that $[y,\overline{\chi}]_x$ $(x\in\Lambda)$ denotes the Wronskian of the solutions $y=y(x,\lambda)$ and $\chi=\chi(x,\lambda)$ of (1.1).

For $y\in D$, let us consider the following boundary and transmission conditions

$$y(c_0)\cos\alpha+y^{[1]}(c_0)\sin\alpha=0, \tag{1.2}$$

$$[y,z]_{c_{n+1}}-h[y,u]_{c_{n+1}}=0, \tag{1.3}$$

$$y(c_m-)=\kappa_m y(c_m+), \tag{1.4}$$

$$y^{[1]}(c_m-)=\kappa_m' y^{[1]}(c_m+), \tag{1.5}$$

where α,κ_m and κ_m' are real numbers with $\kappa_m\kappa_m'>0$, $m=1,2,\dots,n$, and h is some complex number such that $h=\Re h+i\Im h$ with $\Im h>0$.

The spectral analysis and some properties of the regular symmetric (selfadjoint) boundary value transmission problems (BVTPs) have been studied in [1,10,11,14]. On the other hand singular dissipative boundary value transmission and boundary value problems have been investigated in [3,4].

It is known that all eigenvalues of the dissipative operators lie in the closed upper half-plane but this analysis is so weak. To complete the analysis of a dissipative operator there are some methods. One of them is about Livšic's theorem. In this paper, using Livšic's theorem the spectral analysis of the problem (1.1)–(1.5) is investigated. Further it should be noted that the results in this paper are the generalization of the results of [4].