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A multiplicative property characterizes quasinormal composition operators in L^2-spaces

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ABSTRACT

A densely defined composition operator in an L^2-space induced by a measurable transformation ϕ is shown to be quasinormal if and only if the Radon–Nikodym derivatives h_{ϕ^n} attached to powers ϕ^n of ϕ have the multiplicative property: $\mathsf{h}_{\phi^n} = \mathsf{h}_\phi^n$ almost everywhere for $n = 0, 1, 2, \ldots$.

1. Introduction

Composition operators (in L^2-spaces over σ-finite measure spaces) play an essential role in ergodic theory. They are also interesting objects of operator theory. The foundations of the theory of bounded composition operators are well-developed. In particular, the questions of their boundedness, normality, quasinormality, subnormality, seminormality etc. were answered (see e.g., [21,19,26,12,15,16,9,11,22,6] for the general approach and [10,17,23,8,24] for special classes of operators; see also the monograph [22]).

As opposed to the bounded case, the theory of unbounded composition operators is at a rather early stage of development. There are few papers concerning this issue. Some basic facts about unbounded composition operators can be found in [7,13,4]. In a recent paper [5], we gave the first ever criterion for subnormality of unbounded densely defined composition operators, which states that if such an operator admits a measurable family of probability measures that satisfy the consistency condition (see (CC)), then it is subnormal (cf. [5, Theorem 9]). The aforesaid criterion becomes a full characterization of subnormality in the bounded case. Recall that the celebrated Lambert's characterization of subnormality of bounded composition operators (cf. [15]) is no longer true for unbounded ones (see [14, Theorem 4.3.3] and [4, Section 11]). It turns out that the consistency condition is strongly related to quasinormality.

Quasinormal operators, which were introduced by A. Brown in [3], form a class of operators which is properly larger than that of normal operators, and properly smaller than that of subnormal operators (see [3, Theorem 1] and [25, Theorem 2]). It was A. Lambert who noticed that if C_ϕ is a bounded quasinormal composition operator with a surjective symbol ϕ, then the Radon–Nikodym derivatives h_{ϕ^n}, $n = 0, 1, 2, \ldots$, (see (2.1)) have the following multiplicative property (cf. [15, p. 752]):

$$\mathsf{h}_{\phi^n} = \mathsf{h}_\phi^n \quad \text{almost everywhere for } n = 0, 1, 2, \ldots.$$

The aim of this article is to show that the above completely characterizes quasinormal composition operators regardless of whether they are bounded or not, and regardless of whether ϕ is surjective or not (cf. Theorem 3.1). The proof of

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this characterization depends on the fact that a quasinormal composition operator always admits a special measurable family of probability measures which satisfy the consistency condition (CC). This leads to yet another characterization of quasinormality (see condition (iii) of Theorem 3.1).

2. Preliminaries

We write $\mathbb{C}$ for the field of all complex numbers and denote by $\mathbb{R}_+$, $\mathbb{Z}_+$ and $\mathbb{N}$ the sets of nonnegative real numbers, nonnegative integers and positive integers, respectively. Set $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$. Given a sequence $\{\Delta_n\}_{n=1}^\infty$ of sets and a set Δ such that $\Delta_n \subseteq \Delta_{n+1}$ for every $n \in \mathbb{N}$, and $\Delta = \bigcup_{n=1}^\infty \Delta_n$, we write $\Delta_n \nearrow \Delta$ (as $n \to \infty$). The characteristic function of a set Δ is denoted by χ_Δ (it is clear from the context on which set the function χ_Δ is defined).

The following lemma is a direct consequence of [18, Proposition I-6-1] and [1, Theorem 1.3.10]. It will be used in the proof of Theorem 3.1.

Lemma 2.1. *Let $\mathscr{P}$ be a semi-algebra of subsets of a set X and ρ_1, ρ_2 be finite measures*[1] *defined on the σ-algebra generated by $\mathscr{P}$ such that $\rho_1(\Delta) = \rho_2(\Delta)$ for all $\Delta \in \mathscr{P}$. Then $\rho_1 = \rho_2$.*

Let A be a linear operator in a complex Hilbert space $\mathcal{H}$. Denote by $\mathcal{D}(A)$ and A^* the domain and the adjoint of A (in case it exists). If A is closed and densely defined, then A has a (unique) *polar decomposition* $A = U|A|$, where U is a partial isometry on $\mathcal{H}$ such that the kernels of U and A coincide and $|A|$ is the square root of A^*A (cf. [2, Section 8.1]). A densely defined linear operator A in $\mathcal{H}$ is said to be *quasinormal* if A is closed and $U|A| \subseteq |A|U$, where $A = U|A|$ is the polar decomposition of A. We refer the reader to [3] and [25] for basic information on bounded and unbounded quasinormal operators, respectively.

Throughout the paper $(X, \mathscr{A}, \mu)$ will denote a σ-finite measure space. We shall abbreviate the expressions "almost everywhere with respect to μ" and "for μ-almost every x" to "a.e. $[\mu]$" and "for μ-a.e. x", respectively. As usual, $L^2(\mu) = L^2(X, \mathscr{A}, \mu)$ denotes the Hilbert space of all square integrable complex functions on X with the standard inner product. Let $\phi\colon X \to X$ be an $\mathscr{A}$*-measurable* transformation of X, i.e., $\phi^{-1}(\Delta) \in \mathscr{A}$ for all $\Delta \in \mathscr{A}$. Denote by $\mu \circ \phi^{-1}$ the measure on $\mathscr{A}$ given by $\mu \circ \phi^{-1}(\Delta) = \mu(\phi^{-1}(\Delta))$ for $\Delta \in \mathscr{A}$. We say that ϕ is *nonsingular* if $\mu \circ \phi^{-1}$ is absolutely continuous with respect to μ. If ϕ is a nonsingular transformation of X, then the map $C_\phi\colon L^2(\mu) \supseteq \mathcal{D}(C_\phi) \to L^2(\mu)$ given by

$$\mathcal{D}(C_\phi) = \{f \in L^2(\mu) : f \circ \phi \in L^2(\mu)\} \quad \text{and} \quad C_\phi f = f \circ \phi \quad \text{for } f \in \mathcal{D}(C_\phi),$$

is well-defined (and vice versa). Call such C_ϕ a *composition operator*. Note that every composition operator is closed (see e.g., [4, Proposition 3.2]). If ϕ is nonsingular, then by the Radon–Nikodym theorem there exists a unique (up to sets of measure zero) $\mathscr{A}$-measurable function $\mathsf{h}_\phi\colon X \to \overline{\mathbb{R}}_+$ such that

$$\mu \circ \phi^{-1}(\Delta) = \int_\Delta \mathsf{h}_\phi \, d\mu, \quad \Delta \in \mathscr{A}. \tag{2.1}$$

It is well-known that C_ϕ is densely defined if and only if $\mathsf{h}_\phi < \infty$ a.e. $[\mu]$ (cf. [7, Lemma 6.1]), and $\mathcal{D}(C_\phi) = L^2(\mu)$ if and only if $\mathsf{h}_\phi \in L^\infty(\mu)$ (cf. [19, Theorem 1]). Given $n \in \mathbb{N}$, we denote by ϕ^n the n-fold composition of ϕ with itself; ϕ^0 is the identity transformation of X. Note that if ϕ is nonsingular and $n \in \mathbb{Z}_+$, then ϕ^n is nonsingular and thus h_{ϕ^n} makes sense. Clearly $\mathsf{h}_{\phi^0} = 1$ a.e. $[\mu]$.

Suppose that $\phi\colon X \to X$ is a nonsingular transformation such that $\mathsf{h}_\phi < \infty$ a.e. $[\mu]$. Then the measure $\mu|_{\phi^{-1}(\mathscr{A})}$ is σ-finite (cf. [4, Proposition 3.2]). Hence, by the Radon–Nikodym theorem, for every $\mathscr{A}$-measurable function $f\colon X \to \overline{\mathbb{R}}_+$ there exists a unique (up to sets of measure zero) $\phi^{-1}(\mathscr{A})$-measurable function $\mathsf{E}(f)\colon X \to \overline{\mathbb{R}}_+$ such that

$$\int_{\phi^{-1}(\Delta)} f \, d\mu = \int_{\phi^{-1}(\Delta)} \mathsf{E}(f) \, d\mu, \quad \Delta \in \mathscr{A}. \tag{2.2}$$

We call $\mathsf{E}(f)$ the *conditional expectation* of f with respect to $\phi^{-1}(\mathscr{A})$ (see [4] for recent applications of the conditional expectation in the theory of unbounded composition operators; see also [20] for the foundations of the theory of probabilistic conditional expectation). It is well-known that

$$\text{if } 0 \leqslant f_n \nearrow f \text{ and } f, f_n \text{ are } \mathscr{A}\text{-measurable, then } \mathsf{E}(f_n) \nearrow \mathsf{E}(f), \tag{2.3}$$

where $g_n \nearrow g$ means that for μ-a.e. $x \in X$, the sequence $\{g_n(x)\}_{n=1}^\infty$ is monotonically increasing and convergent to $g(x)$.

Now we state three results, each of which will be used in the proof of Theorem 3.1. The first one provides a necessary and sufficient condition for the Radon–Nikodym derivatives h_{ϕ^n}, $n \in \mathbb{N}$, to have the following semigroup property.

Lemma 2.2 ([4, Lemma 9.1])**.** *If ϕ is a nonsingular transformation of X such that $\mathsf{h}_\phi < \infty$ a.e. $[\mu]$ and $n \in \mathbb{N}$, then the following two conditions are equivalent:*

(i) $\mathsf{h}_{\phi^{n+1}} = \mathsf{h}_{\phi^n} \cdot \mathsf{h}_\phi$ *a.e.* $[\mu]$,
(ii) $\mathsf{E}(\mathsf{h}_{\phi^n}) = \mathsf{h}_{\phi^n} \circ \phi$ *a.e.* $[\mu|_{\phi^{-1}(\mathscr{A})}]$.

The second result is a basic description of quasinormal composition operators.

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[1] All measures considered in this paper are assumed to be positive.