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Existence of periodic travelling wave solutions of non-autonomous reaction–diffusion equations with lambda–omega type

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ABSTRACT

Periodic travelling wave solutions of reaction–diffusion equations were studied by many authors. The λ–ω type reaction–diffusion system is a notable special model that admits explicit periodic travelling wave solutions and was introduced by Kopell and Howard in 1973. There are now similar systems which are investigated by means of autonomous dynamics. In contrast, there are few papers which are concerned with non-autonomous cases. For this reason, we apply Mawhin's continuation theorem to derive the existence of periodic travelling wave solutions for non-autonomous λ–ω systems, and we describe the 'disappearance' of periodic travelling wave solutions under special situations. Our main result is also illustrated by examples.

1. Introduction

Periodic travelling wave solutions of reaction–diffusion equations were studied by many authors. In particular, starting from the paper by Kopell and Howard [8], there are now many (see e.g. [1,8,12,3,9,2,6,11,5,4,7] and the references therein) that study such solutions of models with two interacting components:

$$\begin{cases} \dfrac{\partial U}{\partial t} = D_u \nabla^2 U + F(U,V) \\ \dfrac{\partial V}{\partial t} = D_v \nabla^2 V + G(U,V) \end{cases}$$

where U, V are state values of the interacting components, and D_u and D_v are positive constants. However, there are few papers which are concerned with non-autonomous reaction–diffusion equations. For this reason, we discuss here the non-autonomous λ–ω model

$$\begin{cases} \dfrac{\partial U}{\partial t} = D \nabla^2 U + A\lambda(r)U - B\omega(r)V \\ \dfrac{\partial V}{\partial t} = D \nabla^2 V + B\omega(r)U + A\lambda(r)V \end{cases} \tag{1}$$

where

(A1) D is a positive constant and $r(z,t) = U^2(z,t) + V^2(z,t)$ for $z \in \mathbf{R}$ and $t \in \mathbf{R}$;
(A2) λ and ω are continuous functions on $[0,\infty)$, and there exists $\rho > 0$ such that $\omega(\rho) \neq 0$, $\lambda(\rho) = 0$, and $\lambda(\eta) < 0$ for $\eta > \rho$; and
(A3) A and B are continuous functions on $\mathbf{R}^2$ with $A(z,t) > 0$ on a.e. $\mathbf{R}^2$.

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Our model (1) is an extension of the well known (autonomous) λ–ω system introduced by Kopell and Howard in [8]. Indeed, we introduce additional coefficient functions $A = A(z, t)$ and $B = B(z, t)$ which may be different from 1. Other general λ–ω systems are discussed in different sources (see e.g. [6]). For instance, the authors in [8] applied the Hopf bifurcation theorem to illustrate the existence of one parameter family of periodic travelling solutions under the case that A and B are constant functions. We further assume that

$$\lambda(r) = 1 - r \quad \text{and} \quad \omega(r) = \omega_0 - \omega_1 r,$$

where $\omega_0 > \omega_1 > 0$ and $r \geq 0$. As explained in [11] and others, the corresponding autonomous system has an 'explicit' family of 'periodic travelling wave solutions' of the simple form

$$(U(x,t), V(x,t)) = \left(r^* \cos\left(\omega\left(r^*\right) t \pm \lambda\left(r^*\right)^{1/2} x\right),\ R^* \sin\left(\omega\left(r^*\right) t \pm \lambda\left(r^*\right)^{1/2} x\right)\right),$$

where $r^* \in (0, 1)$ is a parameter. In this paper, we provide a sufficient condition to demonstrate the existence of periodic travelling solutions of non-autonomous reaction–diffusion systems. Our tool is Mawhin's continuation theorem (see e.g. [10,1]). In addition, we are able to show that as ρ defined by (A2) approaches zero, periodic travelling wave solutions of (1) will gradually disappear and approach the trivial solution (see Lemma 2.3).

To facilitate further discussions, we first recall some standard notations and Mawhin's continuation theorem. Let T be a positive number. The notations C_T and C_T'' denote respectively the set of all continuous functions on $\mathbf{R}$ with least period T, and the set of all twice continuously differentiable functions on $\mathbf{R}$ with least period T. For any $u \in C_T$, the number $\widetilde{u}$ denotes the average

$$\widetilde{u} = \frac{1}{T}\int_0^T u(s)ds.$$

Definition 1.1. Let $q \in \mathbf{R}$, $p \geq 0$ with $q \neq 0$ and $T > 0$. Then $(U(z,t), V(z,t))$ is said to be a travelling wave with speed $p/|q|$ if it is of the form

$$(U(z,t), V(z,t)) = (u(pt - qz), v(pt - qz)), \tag{2}$$

where u and v are twice continuously differentiable on $\mathbf{R}$. Furthermore, if (u, v) is T-periodic, then (u, v) is called a T-periodic travelling wave with speed $p/|q|$.

The following is Mawhin's continuation theorem (which can be found in [10,1]).

Theorem 1.1 (*Mawhin Continuation Theorem*)**.** *Let X and Y be two normal spaces, $\Omega \subset X$ be open and bounded, $L : D(L) \cap X \to Y$ be a Fredholm linear function where $D(L)$ is the domain of L, P and Q be projections defined on X and Y respectively, and $J : \ker L \longrightarrow \operatorname{Im} Q$ is an isomorphism. Assume that $\ker L = \operatorname{Im} P$ and $N : \overline{\Omega} \to Y$ is a L-compact operator, that is $L_P^{-1}(I - Q)N$ is compact on $\overline{\Omega}$, QN is continuous, and $QN(\overline{\Omega})$ is a bounded set in Y where $L_P^{-1} : \operatorname{Im} L \longrightarrow \ker P \cap D(L)$ denotes the inverse function of L on $\ker P$. Suppose*

(i) *$Lu \neq \varepsilon Nu$ for each $\varepsilon \in (0, 1)$ and $u \in \partial\Omega \cap (D(L) \setminus \ker L)$;*
(ii) *$QNu \neq 0$ for $u \in \partial\Omega \cap \ker L$; and*
(iii) *$d(J^{-1}QN, \Omega \cap \ker L, 0) \neq 0$.*

Then $L - N$ has at least one zero in $D(L) \cap \overline{\Omega}$.

2. Main results

Let $p \geq 0$ and $q \in \mathbf{R}$ with $q \neq 0$ be given. Throughout this paper, we assume that there exist $a, b \in C_T$ such that $A(z,t) = a(pt - qz)$ and $B(z,t) = b(pt - qz)$ for $z, t \in \mathbf{R}$. We can observe that if the system

$$\begin{cases} q^2Dx'' - px' = -a\lambda(x^2 + y^2)x + b\omega(x^2 + y^2)y \\ q^2Dx'' - py' = -b\omega(x^2 + y^2)x - a\lambda(x^2 + y^2)y \end{cases} \tag{3}$$

has a solution (x, y), then (U, V) defined by

$$U(z,t) = x(pt - qz) \quad \text{and} \quad V(z,t) = y(pt - qz) \tag{4}$$

is a travelling wave solution of (1) with speed $p/|q|$. Conversely, if (U, V) is a travelling wave solution of (1) with speed $p/|q|$, then (x, y) that satisfies (4) is a solution of (3). By this observation, we shall prove that the system (3) has a periodic solution.

Let $T > 0$ be given. We define sets X and Y by

$$X = Y = C_T \times C_T.$$

Then X and Y are normal spaces with a norm $\|(u, v)\| = \max_{t\in[0,T]} (|u(t)| + |v(t)|)$. Let $D(L) = X \cap \left(C_T'' \times C_T''\right)$. We define an operator L on $D(L)$ by

$$L(u, v) = \left(q^2Du'' - pu', q^2Dv'' - pv'\right).$$