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Inverse closed ultradifferential subalgebras

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ABSTRACT

In previous work we have shown that classical approximation theory provides methods for the systematic construction of inverse-closed smooth subalgebras. Now we extend this work to treat inverse-closed subalgebras of ultradifferentiable elements. In particular, Carleman classes and Dales–Davie algebras are treated. As an application the result of Demko, Smith and Moss, and Jaffard on the inverse of a matrix with exponential decay is obtained within the framework of a general theory of smoothness.

1. Introduction

We describe new methods to generate a smooth inverse-closed subalgebra of a given Banach algebra $\mathcal{A}$ and to characterize this subalgebra by approximation properties and by weights. Recall that a subalgebra $\mathcal{B}$ of $\mathcal{A}$ is inverse-closed in $\mathcal{A}$, if

every $b \in \mathcal{B}$ that is invertible in $\mathcal{A}$ is actually invertible in $\mathcal{B}$.

A prototype of an inverse-closed subalgebra is the Wiener algebra of absolutely convergent Fourier series, which is inverse-closed in the algebra of continuous functions on the torus. Another example is the algebra $C^1(\mathbb{T})$ of continuously differentiable functions on the torus; the proof that $C^1(\mathbb{T})$ is inverse-closed in $C(\mathbb{T})$ is essentially the quotient rule of classical analysis.

Many methods for the construction of inverse-closed subalgebras are based on generalizations of this simple smoothness principle. In the context of Banach algebras, derivatives are replaced by derivations. The Leibniz rule for derivations implies that their domain is a Banach algebra, and by the symmetry of $\mathcal{A}$ the domain is inverse-closed in $\mathcal{A}$; see [15].

A more refined concept of smoothness can be developed, if $\mathcal{A}$ is invariant under the bounded action of a d-dimensional automorphism group. In this case algebras of Bessel–Besov type can be defined, and the properties of the group action imply that the spaces defined form inverse-closed subalgebras of $\mathcal{A}$; see [20].

A different approach to smoothness is by approximation using approximation schemes adapted to the algebra multiplication. This line of research, initiated by Almira and Luther [2,3], yields Banach algebras of approximation spaces that are inverse-closed in $\mathcal{A}$, if $\mathcal{A}$ is symmetric [15].

Moreover, if $\mathcal{A}$ is invariant under the action of the translation group and the approximation scheme consists of the *bandlimited elements* of $\mathcal{A}$, we obtain Jackson–Bernstein theorems that identify approximation spaces of polynomial order with Besov spaces.

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All of the above has been carried out in two previous publications [15,20] for smoothness spaces of finite order. Now we use the same principles to construct inverse-closed subalgebras of ultradifferentiable elements.

Classes of Carleman type are defined by growth conditions on the norms of higher derivations in the same way as for functions, and we obtain a characterization of inverse-closed Carleman classes by adapting a proof of Siddiqi [31]. If the growth of the derivations satisfies the condition (M2′) of Komatsu, then an alternative description of the Carleman classes as union of weighted spaces or approximation spaces is available.

Whereas Carleman algebras are inductive limits of Banach spaces we can also define *Banach algebras* of ultradifferentiable elements derived from a given Banach algebra. The construction generalizes an approach used by Dales and Davie [7] for functions defined on perfect subsets of the complex plane, so we call the resulting Banach algebras *Dales–Davie algebras*. A result of Honary and Abtahi [1] on inverse-closed Dales–Davie algebras of functions can be adapted to the noncommutative situation (Theorem 32).

The general theory has applications to Banach algebras of matrices with off-diagonal decay. The formal commutator $\delta(A) = [X, A], X = 2\pi i \mathrm{Diag}((k)_{k\in\mathbb{Z}})$, is a derivation on $\mathcal{B}(\ell^2)$, and its domain defines an algebra of matrices with off-diagonal decay that is inverse-closed in $\mathcal{B}(\ell^2)$ [15, 3.4]. The translation group acts boundedly on $\mathcal{B}(\ell^2)$ by conjugation with the modulation operator $M_t = \mathrm{Diag}(e^{2\pi ik\cdot t})_{k\in\mathbb{Z}^d}$,

$$\chi_t(A) = M_tAM_{-t} = \sum_{k\in\mathbb{Z}^d} \hat{A}(k)e^{2\pi ik\cdot t} \quad \text{for } t \in \mathbb{R}^d, \tag{1}$$

where $\hat{A}(k)$ is the kth side diagonal of A,

$$\hat{A}(k)(l, m) = \begin{cases} A(l, m), & l - m = k, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

In [15,20] the theory of smooth and inverse-closed subalgebras has been applied to describe Banach algebras of matrices with off-diagonal decay.

The approximation theoretic characterization of Carleman classes of Gevrey type on $\mathcal{B}(\ell^2)$ yields a new proof of a result of Demko, Smith and Moss [10].

Theorem 1. *If $A \in \mathcal{B}(\ell^2)$ with $|A(k, l)| \le Ce^{-\gamma|k-l|}$ for constants $C, \gamma > 0$ and all $k, l, \in \mathbb{Z}^d$, and if $A^{-1} \in \mathcal{B}(\ell^2)$, then there exist $C', \gamma' > 0$ such that*

$$|A^{-1}|(k, l) \le C'e^{-\gamma'|k-l|} \quad \textit{for all } k, l \in \mathbb{Z}^d.$$

In some instances, Dales–Davie algebras of matrices can be identified with known Banach algebras of matrices, e.g. if $\mathcal{C}^1_{v_0}$ consists of matrices with norm

$$\|A\|_{\mathcal{C}^1_{v_0}} = \sum_{k\in\mathbb{Z}^d} \sup_{l\in\mathbb{Z}^d} |A(l, l - k)|,$$

then $D^1_M(\mathcal{C}^1_{v_0})$ is a weighted form of this algebra for a submultiplicative weight v_M associated to M; see Section 4.

The organization of the paper is as follows. First we recall some facts from the theory of Banach algebras and review results of [15,20] on inverse-closed subalgebras of a given Banach algebra defined by derivations, automorphism groups, and approximation spaces. In Section 3, after treating C^∞ classes, ultradifferentiable classes of Carleman type are introduced, and necessary and sufficient conditions on their inverse-closedness are given. Carleman classes satisfying axiom (M2′) of Komatsu are characterized by approximation and weight conditions. As an application we generalize the result of Demko [10] on the inverses of matrices with exponential off-diagonal decay. The results on the inverse-closedness of Dales–Davie algebras are treated in Section 4. In Section 5 some applications to matrix algebras with off-diagonal decay are given. In the Appendix a combinatorial lemma on the iterated quotient rule is proved.

2. Preliminaries

2.1. Notation

The cardinality of a finite set A is $|A|$. The *d-dimensional torus* is $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. The symbol $\lfloor x\rfloor$ denotes the greatest integer smaller than or equal to the real number x. Positive constants will be denoted by C, C', C_1, c, etc., where the same symbol might denote different constants in each equation.

We use the standard multi-index notation. Multi-indices are denoted by Greek letters and are d-tuples of nonnegative integers. The degree of $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ is $|\alpha| = \sum_{j=1}^d \alpha_j$, and $D^\alpha f(x) = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} f(x)$ is the partial derivative. The inequality $\beta \le \alpha$ means that $\beta_j \le \alpha_j$ for all indices j. The p-norm on $\mathbb{C}^d$ is denoted by $|x|_p = \left(\sum_{k=1}^d |x(k)|^p\right)^{1/p}$.

A submultiplicative weight on $\mathbb{Z}^d$ is a positive function $v : \mathbb{Z}^d \to \mathbb{R}$ such that $v(0) = 1$ and $v(x + y) \le v(x)v(y)$ for $x, y \in \mathbb{Z}^d$. The standard polynomial weights are $v_r(x) = (1 + |x|)^r$ for $r \ge 0$. The weighted spaces $\ell^p_w(\mathbb{Z}^d)$ are defined by the