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Finite-time consensus on strongly convex balls of Riemannian manifolds with switching directed communication topologies

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ABSTRACT

It is known that a consensus problem on any connected complete Riemannian manifold can be transformed into the one on its strongly convex balls via the compression–decompression along geodesics. From the viewpoint of interior metrics, this paper mainly provides a consensus protocol for strongly convex geodesic balls, in which the communication can be switching and directed. With the aid of nonsmooth analysis tools on Riemannian manifolds, our analysis shows that all dynamical points involved can achieve consensus in finite time. Meanwhile, the corresponding global algorithm is given, with its application to the consensus problem of rotation attitudes, as well as a case simulation, to demonstrate and verify our proposed techniques.

1. Introduction

It is well-known that Riemannian manifolds have various topologies, which makes us difficult to design the distributed control systems on Riemannian manifolds. However, the prototypes of Riemannian manifolds range from Euclidean surfaces to Lie groups, matrix spaces, operator spaces, etc., which ensures that Riemannian manifolds have broad application in practice, especially in modeling motion spaces and data spaces, such as rotation group $SO(3)$ for 3D rotation attitude, torus $\mathbb{S} \times \mathbb{S}$ for double planar pendulums, elliptic orbits for satellites and unit sphere $\mathbb{S}^2$ in image processing, etc. It is clear that there exist distributed control problems, and some of them are just consensus problems. In view of the fact that finite-time convergence plays an important role in practice, the paper studies the finite-time consensus problems on Riemannian manifolds.

For solving consensus problems on manifold spaces, consensus algorithms for homogeneous manifolds [27], symmetric spaces [29] and special orthogonal groups [26] were proposed from the viewpoint of exterior metrics. However, these works failed to discuss the finite-time consensus problems. Consensus approaches with interior metrics were also reported, such as consensus protocols for the rotation group [33] and consensus algorithms for general Riemannian manifolds [32,7]. However, all communication networks in these works are undirected, although one of them explored the finite-time consensus problems.

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In fact, various finite-time design methods in the context of Euclidean spaces have been proposed in the past years, including normalized and signed gradient systems [9], binary control [6], monotonic function [16], observer-based control [39], and containment control [36], etc. Meanwhile, lots of finite-time consensus algorithms for networked motion objects were also put forth, such as rigid spacecrafts [10], Euler–Lagrangian systems [37] and nonholonomic mobile robots [21], etc. However, the Euclidean metrics or the exterior metrics of some Riemannian manifolds were used there. Access Refs. [34,38,35,11,23] and some others for the recent development of design methods to general consensus problems on Euclidean spaces.

In our previous work [7], we have shown the technique that a consensus protocol problem on any connected complete Riemannian manifold can be transformed into the one on its strongly convex balls, via the compression–decompression along geodesics. This motivates us to study the finite-time consensus problems restricted on strongly convex balls. Designing from the viewpoint of interior metrics, and analyzing with nonsmooth analysis tools on Riemannian manifolds, the paper mainly shows the following result.

Main Theorem. *Let M be a finite-dimensional Riemannian manifold, and $r_o(M)$ be its convexity radius at $o \in M$. Set $B_o = \{y \in M : \varrho_o(y) < r\}$, $r \in (0, r_o(M)/3)$. Consider the following dynamical system:*

$$\dot{x}_i = \frac{\exp_{x_i}^{-1} x_{i^*}}{\| \exp_{x_i}^{-1} x_{i^*} \|}, \quad i \in \bar{n}, \tag{1}$$

with a communication topology graph $G_{\sigma(t)} \in \mathbb{G}^n$, where each $\exp_{x_i}^{-1}$ is the reverse of the exponential map $\exp_{x_i} : T_{x_i}M \to M$, and each $i^* = i^*(t) \in N_i(G_{\sigma(t)})$ satisfies the condition that $\varrho(x_i, x_j) \le \varrho(x_i, x_{i^*})$ holds for any $j \in N_i(G_{\sigma(t)})$. Assume that all initial states $x_i(0) \in B_o$. Then, if the graph $G_{\sigma(t)}$ has a center and each set $N_i(G_{\sigma(t)})$ is nonempty, then all $x_i(t)$ in (1) achieve consensus on the ball B_o in finite time.

Here, the communication can be directed and dynamically switching, which distinguishes it from those works using interior metrics to design consensus algorithms for Riemannian manifolds. However, the main theorem just provides a local consensus algorithm. In other words, if one designs the consensus algorithm for Riemannian manifolds according to this theorem, one must firstly ensure the initial states of all dynamical points in a strongly convex ball B_o. Thanks to the compression and decompression functions [7] providing a mapping between the dynamical system (1) and some one on the manifold, with the merit that the initial states of the latter can be taken randomly from the manifold, one can extend the local algorithm to be a global one stated as follows.

Global Finite-Time Consensus Algorithm. Continue the main theorem. Further assume that M is connected and complete, and $\Phi : M \to B_o$ is a compression function with its decompression function $\Psi : B_o \to M$. Then, given any n points $q_1^0, \ldots, q_n^0 \in M$, all dynamical points $\Psi(p_1(t)), \ldots, \Psi(p_n(t))$ with the respective initial states $q_1^0, \ldots, q_n^0$ achieve consensus in finite time, if each $p_i(0) = \Phi(q_i^0)$.

Our originality mainly focuses on the local finite-time consensus algorithm in the main theorem. Together with the above global one, these consensus schemes can provide common solutions to the consensus problems arising from various nonlinear spaces or abstract spaces belonging to connected complete Riemannian manifolds, where the communication can be directed and dynamically switching. In view of the fact that the consensus problems of rotation attitudes play an important role in practice, the global algorithm is also applied to such practical problems, together with a case simulation, to demonstrate and verify our proposed techniques.

The rest of the paper is organized as follows. In the next section, some preliminaries and notations about Riemannian geometrical objects and communication topology graphs are provided. Section 3 studies the nonsmooth analysis tools on Riemannian manifolds, aiming for providing the analysis foundations for the next section. Our main theorem is proved in Section 4, based on the behavior analysis of global invariance and individual dynamics. Before concluding the paper, the global algorithm is applied to the consensus problem of rotation attitudes, together with a case simulation.

2. Preliminaries and notations

In the section, some preliminaries and notations about Riemannian geometrical objects and communication topology graphs are provided.

2.1. Sets and spaces

For the Riemannian manifold M, say that an open set $U \subset M$ is strongly convex if any $p, q \in U$ can be joined by exactly one geodesic of distance length $\varrho_M(p, q)$ which belongs entirely to U. Following Refs. [17,24], every $e \in M$ possesses a convexity radius $r_e(M)$ such that the geodesic ball $\{q \in M : \varrho_M(e, q) < r\}$ is strongly convex with $0 < r < r_e(M)$. Intuitively, the convexity radius $r_e(M)$ may be small in general; however, some Riemannian manifolds in common use behave differently. For example, $r_e(\mathbb{S}^2) = \pi/2$ and $r_e(SO(m)) \ge \pi/2$ at any point e. In fact, any compact Riemannian manifold N has the positive