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Some results for impulsive problems via Morse theory

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ABSTRACT

We use Morse theory to study impulsive problems. First we consider asymptotically piecewise linear problems with superlinear impulses, and prove a new existence result for this class of problems using the saddle point theorem. Next we compute the critical groups at zero when the impulses are asymptotically linear near zero, in particular, we identify an important resonance set for this problem. As an application, we finally obtain a nontrivial solution for asymptotically piecewise linear problems with impulses that are asymptotically linear at zero and superlinear at infinity. Our results here are based on the simple observation that the underlying Sobolev space naturally splits into a certain finite dimensional subspace where all the impulses take place and its orthogonal complement that is free of impulsive effects.

1. Introduction

Impulsive problems arise naturally in studies of evolutionary processes that involve abrupt changes in the state of the system, triggered by instantaneous perturbations called impulses. Examples include games where players can affect the game only at discrete instants (see Chikrii, Matychyn, and Chikrii [3]), two person zero sum games with separated impulsive dynamics (see Crück, Quincampoix, and Saint-Pierre [5]), pulse vaccination strategy (see Stone, Shulgin, and Agur [11]), and optimal impulsive harvesting (see Zhang, Shuai, and Wang [14]). Classical approaches to such problems include fixed point theory (see, e.g., Lin and Jiang [8]) and the method of upper and lower solutions (see, e.g., Liu and Guo [9]). More recently, variational methods have been widely used to study impulsive problems (see, e.g., Tian and Ge [12], Nieto and O'Regan [10], Zhou and Li [16], Zhang and Yuan [15], Zhang and Li [13], Bai and Dai [1], Han and Wang [7], and Gong, Zhang, and Tang [6]).

In this paper we use Morse theory to study impulsive problems. First we consider asymptotically piecewise linear problems with superlinear impulses. Although asymptotically piecewise linear nonlinearities are quite natural in this setting, they do not seem to have been studied in the literature. We will prove a new existence result for this class of problems using the saddle point theorem. Next we compute the critical groups at zero when the impulses are asymptotically linear near zero. In particular, we will identify an important resonance set for this problem. The effect of impulses on critical groups has not been studied previously, to the best of our knowledge. As an application, we finally obtain a nontrivial solution for asymptotically piecewise linear problems with impulses that are asymptotically linear at zero and superlinear at infinity. Our results here are based on the simple observation that the underlying Sobolev space naturally splits into a certain finite dimensional subspace where all the impulses take place and its orthogonal complement that is free of impulsive effects.

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Let m be a positive integer, let $0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = 1$, and consider the impulsive problem

$$\begin{cases} -u'' = f(x, u), \quad x \in (0, 1) \setminus \{x_1, \ldots, x_m\} \\ u(0) = u(1) = 0, \qquad u(x_j^+) = u(x_j^-), \quad j = 1, \ldots, m \\ u'(x_j^+) = u'(x_j^-) - \imath_j(u(x_j)), \quad j = 1, \ldots, m, \end{cases} \tag{1.1}$$

where f is a Carathéodory function on $(0, 1) \times \mathbb{R}$,

$$u(x_j^\pm) = \lim_{\substack{x \to x_j \\ x \gtrless x_j}} u(x), \qquad u'(x_j^\pm) = \lim_{\substack{x \to x_j \\ x \gtrless x_j}} u'(x),$$

and $\imath_j$ are continuous functions on $\mathbb{R}$. Denoting by $H_0^1(0, 1)$ the usual Sobolev space with the inner product

$$(u, v) = \int_0^1 u'v',$$

a weak solution of (1.1) is a function $u \in H_0^1(0, 1)$ such that

$$\int_0^1 u'v' = \int_0^1 f(x, u)\, v + \sum_{j=1}^m \imath_j(u(x_j))\, v(x_j) \quad \forall v \in H_0^1(0, 1).$$

Noting that $H_0^1(0, 1)$ is continuously embedded in $C[0, 1]$, we see that weak solutions coincide with the critical points of the C^1-functional

$$\Phi(u) = \frac{1}{2}\int_0^1 (u')^2 - \int_0^1 F(x, u) - \sum_{j=1}^m I_j(u(x_j)), \quad u \in H = H_0^1(0, 1),$$

where

$$F(x, t) = \int_0^t f(x, s)\, ds, \qquad I_j(t) = \int_0^t \imath_j(s)\, ds$$

are the primitives of f and $\imath_j$, respectively.

The closed linear subspace

$$N = \big\{u \in H : u(x_j) = 0,\ j = 1, \ldots, m\big\}$$

is important here since each $I_j(0) = 0$. For $j = 1, \ldots, m$, the mapping $H \to \mathbb{R},\ u \mapsto u(x_j)$ is a bounded linear functional on H and hence there is a unique $w_j \in H$ such that $u(x_j) = \big(u, w_j\big)$ by the Riesz–Frechet representation theorem. In fact,

$$w_j(x) = \begin{cases} (1 - x_j)\, x, & 0 \le x \le x_j \\ x_j\, (1 - x), & x_j \le x \le 1. \end{cases} \tag{1.2}$$

Since x_j are distinct, w_j are linearly independent, so N is the orthogonal complement of the m-dimensional subspace M that they span. Hence we have the orthogonal decomposition

$$H = N \oplus M, \qquad u = v + w,$$

and

$$\Phi(u) = \frac{1}{2}\int_0^1 \big((v')^2 + (w')^2\big) - \int_0^1 F(x, u) - \sum_{j=1}^m I_j(w(x_j)). \tag{1.3}$$

We will make use of this splitting throughout the paper.

By (1.2), each $w \in M$ is affine on the subintervals $[x_{j-1}, x_j]$. Since the space of continuous functions on $[0, 1]$ that are affine on these subintervals and vanish at the endpoints is also m-dimensional, it follows that M is precisely this subspace. Then we also have

$$\max_{x \in [0,1]} |w(x)| = \max_{j=1,\ldots,m} |w(x_j)| \quad \forall w \in M,$$

and this is an equivalent norm on this finite dimensional space.

The subspace N has the decomposition

$$N = \bigoplus_{j=1}^{m+1} N_j, \qquad v = \sum_{j=1}^{m+1} v_j$$