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Quaternionic Cayley transform revisited

F.-H. Vasilescu

Department of Mathematics, University of Lille 1, 59655 Villeneuve d'Ascq, France

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ABSTRACT

A Cayley type transform in the context of quaternions (previously introduced by the author) is revisited. A simplified definition is given and shorter proofs are exhibited. As new results, the range of this transform restricted to a class of unbounded normal operators is described, and an application to a moment problem with constraints is given. The extension problem to normal operators in this new context is also revisited.

0. Introduction

The classical Cayley transform

$$\kappa(t) = \frac{t-i}{t+i} = \frac{t^2-1-2it}{1+t^2}$$

is a bijective map between the real line $\mathbb{R}$ and the set $\mathbb{T} \setminus \{1\}$, where $\mathbb{T}$ is the unit circle in the complex plane $\mathbb{C}$. This formula can be extended to more general situations, as for instance that of (not necessarily bounded) symmetric operators in Hilbert spaces, replacing formally the real variable by such an operator, which yields a homonymic transform whose construction is due to von Neumann [22] (see also [15]). A Cayley type transform may be actually defined for larger classes of operators, which are no longer symmetric, as well as for other objects, in particular for some linear relations (see for example [7]). The usefulness of the Cayley transform, which is largely recognized, has been more recently proved in the study of various properties of algebras of unbounded operators (see [5] or [21]).

In order to find a formula of this type, valid for normal or formally normal operators (see [4]), a possible approach is to consider a quaternionic framework. An attempt to extend this transform using the context of quaternions has been made in [20]. In the present paper, we modify the basic definitions from [20], which allow us to get (in a simpler way) the properties of the quaternionic Cayley transform directly from those of von Neumann's Cayley transform, and refine some results from the quoted work. Unlike in [20], our new construction does not require densely defined operators, which might be useful for potential applications. We mention that we may perform our construction on a sufficiently large class of operators containing some differential operators with matrix coefficients, related to the so-called Dirac operator, as in Example 2.2(2). Nevertheless, we restrict ourselves especially to the class satisfying condition (c) from Lemma 2.4, to have a good control on the inverse quaternionic Cayley transform.

We found it useful to include, in the first section, an approach to the quaternionic Cayley transform in the algebra of quaternions, which is the simplest yet significant case, for a better understanding of the general topics. In addition, some computations from this section are later used.

E-mail address: fhvasil@math.univ-lille1.fr.

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In the second section of this paper, we revisit the construction of the Cayley transform for some operators, in the quaternionic context, as we already mentioned above. The main result from this section (Theorem 2.7) is an extended version of Theorem 2.14 from [20], valid for not necessarily densely defined operators.

We recall that the image of a (not necessarily bounded) self-adjoint operator by the usual Cayley transform is a unitary operator U with the property that $I - U$ is injective, where I is the identity; the converse is also true [15]. Inspired by this property, in the third section of this paper we describe the unitary operators lying in the range of the quaternionic Cayley transform, which are images of some (not necessarily bounded) normal operators. As a matter of fact, it is Theorem 3.7 from this section which is the main (and essentially new) result of the present paper. An example related to this result is given in the last section.

In the fourth section, following the lines of [20] but using the newly introduced framework, we revisit the normal extension problem via the quaternionic Cayley transform. The normal extension problem for a bounded operator goes back to [8,3]. The corresponding problem, stated for an unbounded operator has been first solved in [4]. The oldest result valid for several bounded operators appears in [9], while for the unbounded case for several operators we quote [14,10,11,1], etc.

The main motivation of the introduction of the quaternionic Cayley transform in [20] was to give an answer to such an extension problem, somehow in the spirit of [14] (see also [10,11]), that is, extending simultaneously a pair of symmetric operators to a pair of commuting self-adjoint operators, with applications to some moment problems. At that time, the author (and some other authors as well) was not aware of Coddington's corresponding result from [4]. Although Theorem 2 from [4] and our Theorem 4.7 (via Theorem 3.7 from [20]) are answers to the problem of normal extensions, our method is totally different and has other implications, as those from the third and fifth sections. Moreover, unlike in [4], we can also obtain results for not necessarily densely defined operators (see Corollary 4.8 and Example 4.11). An application of our results is Theorem 4.10, extending Theorem 3.8 from [20], using quite a mild commutativity condition (designated by (C)), and continuing a series of related results appearing in [14,10,11,18], etc. Other applications are to be expected in future work.

Finally, the last section of this work exhibits an example related to Theorem 3.7, showing that some moment problems with constraints may be approached with our methods.

Let us briefly recall the strategy from [20] concerning the type of normal extensions we are dealing with (see also Remark 4.9). Let $\mathcal{D}$ be a dense subspace in a Hilbert space $\mathcal{H}$. Let also T be a densely defined linear operator in $\mathcal{H}$, with the property that T and its adjoint T^* are both defined on $\mathcal{D}$. Writing $T = A + iB$, with $A = (T + T^*)/2$ and $B = (T - T^*)/2i$, and so A and B are symmetric operators on $\mathcal{D}$, we can associate the operator T with the matrix operator

$$Q_T = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}.$$

It is known (see [20, Theorem 3.7]) that T is normal in $\mathcal{H}$ if and only if the operator Q_T is normal in the Hilbert space $\mathcal{H} \oplus \mathcal{H}$. Because our techniques, based on a quaternionic Cayley transform, give conditions to ensure the existence of a normal extension for a matrix operator resembling to Q_T, we can go back to the operator T, which satisfies only some verifiable conditions. In fact, we have such results actually for the case when A and B are symmetric operators defined on a not necessarily dense domain in $\mathcal{H}$. More information in this respect will be given in the fourth section of this work.

Most of the results in this article have been extended to the case of linear relations (see [16]).

Let us finally note that the quaternionic algebra is intimately related also to the spectral theory of pairs of commuting operators (see [19]).

1. Cayley transforms in the algebra of quaternions

In this section, we present an approach to the Cayley transform in the algebra of quaternions.

Consider the 2×2-matrices

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The Hamilton algebra of quaternions $\mathbb{H}$ will be identified with the $\mathbb{R}$-subalgebra of the algebra $\mathbb{M}_2$ of 2×2-matrices with complex entries, generated by the matrices $\mathbf{I}$, $i\mathbf{J}$, $\mathbf{K}$ and $i\mathbf{L}$. The embedding $\mathbb{H} \subset \mathbb{M}_2$ allows us to regard the elements of $\mathbb{H}$ as matrices and to perform some operations in $\mathbb{M}_2$ rather than in $\mathbb{H}$. (The matrices $\mathbf{J}$, $-i\mathbf{K}$ and $\mathbf{L}$, which are called the *Pauli matrices* in mathematical physics, do not belong to $\mathbb{H}$. Nevertheless, the matrices $\mathbf{J}$ and $\mathbf{L}$ play an important role in our development.)

If we put

$$Q(z) = Q(z_1, z_2) = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}$$

for every $z = (z_1, z_2) \in \mathbb{C}^2$, the set $\{Q(z); z \in \mathbb{C}^2\}$ is precisely the algebra of quaternions, because of the decomposition

$$Q(z) = (\mathrm{Re} z_1)\mathbf{I} + i(\mathrm{Im} z_1)\mathbf{J} + (\mathrm{Re} z_2)\mathbf{K} + i(\mathrm{Im} z_2)\mathbf{L}.$$