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Representation and m-term approximation for anisotropic Besov classes[☆]

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ABSTRACT

In this paper, we discuss the best m-term approximation for the anisotropic Besov class of multivariate periodic functions. First we give the representation theorem of the anisotropic Besov class $B^r_{p,\theta}$, then we study the best m-term approximation with respect to orthogonal dictionaries U^d. Moreover, it is established that the asymptotic order of the best m-term approximation can be achieved by a simply greedy algorithm.

1. Introduction

The best m-term approximation is a special form of nonlinear approximation that occurs in many applications including image and signal processing [1]. Nonlinear m-term approximation has been widely studied recently. The basic idea behind nonlinear approximation is that the elements used in the approximation do not come from a fixed linear space but are allowed to depend on the function being approximated.

Denote by X a separable Banach space and by E a system of elements in X such that $\overline{\mathrm{span} E} = X$. Then E is called a dictionary. Denote by $\sum_m(E)$ the collection of all linear combinations of the form

$$g = \sum_{i=1}^{m} c_i g_i, \quad c_i \in R, g_i \in E, i = 1, \dots, m.$$

Note that if we add two elements from $\sum_m(E)$, we will generally need $2m$ terms to represent the sum, so $\sum_m(E)$ is not a linear space.

For a given function class $F \subset X$ and a dictionary E, set

$$\sigma_m(f, E)_X := \inf_{g \in \Sigma_m(E)} \|f - g\|_X,$$
$$\sigma_m(F, E)_X := \sup_{f \in F} \sigma_m(f, E)_X.$$

We call the quantities $\sigma_m(f, E)_X$ and $\sigma_m(F, E)_X$ the best m-term approximation of f and F with regard to E, respectively. Denote by $\mathcal{E}$ a collection E of dictionaries we consider

$$\sigma_m(F, \mathcal{E})_X := \inf_{E \in \mathcal{E}} \sigma_m(F, E)_X.$$

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Thus the quantity $\sigma_m(F, \mathcal{E})_X$ gives the sharp lower bound for the best m-term approximation of a given function class F with regard to any dictionary $E \in \mathcal{E}$.

Now we give the definition of a greedy algorithm for a general basis. Let $\Psi = \{\psi_k\}_{k=1}^{\infty}$ be a basis for X. Represent $f \in X$ in the form

$$f = \sum_{k=1}^{\infty} c_k(f, \Psi)\psi_k.$$

We enumerate the summands in decreasing order

$$\|c_{k_1}(f, \Psi)\psi_{k_1}\| \geq \|c_{k_2}(f, \Psi)\psi_{k_2}\| \geq \cdots$$

and define the mth-greedy algorithm as

$$G_m^X(f, \Psi) := \sum_{i=1}^{m} c_{k_i}(f, \Psi)\psi_{k_i}.$$

Let $1 \leq p \leq \infty$, $T^d := [0, 2\pi)^d$ be the d-dimensional torus, and let $L_p := L_p(T^d)$ be the Banach space of measurable functions $f(x) = f(x_1, \ldots, x_d)$, which is 2π-periodic with respect to each variable. Its norm is defined by

$$\|f\|_p := \|f\|_{L_p(T^d)} := \left\{(2\pi)^{-d} \int_{T^d} |f(x)|^p dx\right\}^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty,$$
$$\|f\|_\infty := \|f\|_{L_\infty(T^d)} := \operatorname{ess}\sup_{x \in T^d} |f(x)| < \infty.$$

Now we give the orthogonal dictionary U^d constructed by Temlyakov (see [6]). We first recall the system $U = \{U_I\}$ in the univariate case. Denote

$$U_n^+(x) := \sum_{k=0}^{2^n-1} e^{ikx} = \frac{e^{i2^n x} - 1}{e^{ix} - 1}, \quad n = 0, 1, 2, \ldots,$$
$$U_{n,k}^+(x) := e^{i2^n x} U_n^+(x - 2\pi k 2^{-n}), \quad k = 0, 1, \ldots, 2^n - 1,$$
$$U_{n,k}^-(x) := e^{-i2^n x} U_n^+(-x + 2\pi k 2^{-n}), \quad k = 0, 1, \ldots, 2^n - 1.$$

It will be more convenient for us to normalize in L_2 the system of functions $\{U_{m,k}^+, U_{n,k}^-\}$ and to enumerate it by dyadic intervals. We write $U_{[0,1)}(x) := 1$,

$$U_I(x) := 2^{-n/2} U_{n,k}^+(x) \quad \text{with } I = \left[\left(k + \frac{1}{2}\right) 2^{-n}, (k+1)2^{-n}\right]$$

and

$$U_I(x) := 2^{-n/2} U_{n,k}^-(x) \quad \text{with } I = \left[k2^{-n}, \left(k + \frac{1}{2}\right) 2^{-n}\right].$$

Denote

$$D_n^+ := \left\{I : I = \left[\left(k + \frac{1}{2}\right) 2^{-n}, (k+1)2^{-n}\right], k = 0, 1, \ldots, 2^n - 1\right\},$$
$$D_n^- := \left\{I : I = \left[k2^{-n}, \left(k + \frac{1}{2}\right) 2^{-n}\right], k = 0, 1, \ldots, 2^n - 1\right\}$$

and

$$D_0^+ = D_0^- = D_0 := [0, 1), \qquad D := \bigcup_{n \geq 1} (D_n^+ \cup D_n^-) \cup D_0.$$

It is easy to check that for any $I, J \in D$, $I \neq J$ we have

$$\langle U_I, U_J \rangle = (2\pi)^{-1} \int_0^{2\pi} U_I(x)\bar{U}_J(x)dx = 0$$

and

$$\|U_I\|_2^2 = 1.$$

We use the notations for $f \in L_1$:

$$f_I := \langle f, U_I \rangle = (2\pi)^{-1} \int_0^{2\pi} f(x)\bar{U}_I(x)dx, \qquad \hat{f}(k) := (2\pi)^{-1} \int_0^{2\pi} f(x)e^{-ikx}dx,$$