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Existence and uniqueness of steady solutions to the magnetohydrodynamic equations of compressible flows

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ABSTRACT

We establish the existence and uniqueness of a strong solution to the steady magnetohydrodynamic equations for the compressible barotropic fluids in a bounded smooth domain with a perfectly conducting boundary, under the assumption that the external force field is small.

1. Introduction and main result

Magnetohydrodynamics (MHD) describes the dynamics of electrically conducting fluids and the theory of the macroscopic interaction of electrically conducting fluids with a magnetic field. The applications of magnetohydrodynamics cover a very wide range. There is a complex interaction between the magnetic and fluid dynamic phenomena, and both hydrodynamic and electrodynamic effects have to be considered. The set of equations are a combination of the compressible Navier–Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism.

In this paper, we shall consider the steady MHD system

$$\begin{cases} \operatorname{div}(\rho\,\mathbf{u}) = 0 & \text{in } \Omega, \\ \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) - \mu\triangle\mathbf{u} - \zeta\nabla\operatorname{div}\mathbf{u} + \nabla p = (\nabla\times\mathbf{H})\times\mathbf{H} + \rho\,\mathbf{f} & \text{in } \Omega, \\ \nabla\times(\nu\nabla\times\mathbf{H}) - \nabla\times(\mathbf{u}\times\mathbf{H}) = \mathbf{0} \quad \text{and} \quad \operatorname{div}\mathbf{H} = 0 & \text{in } \Omega, \\ \mathbf{u}|_{\partial\Omega} = \mathbf{0}, \qquad \mathbf{H}\cdot\mathbf{n}|_{\partial\Omega} = 0, \quad \text{and} \quad (\nabla\times\mathbf{H})\times\mathbf{n}|_{\partial\Omega} = \mathbf{0} & \text{on } \partial\Omega, \\ \displaystyle\int_\Omega \rho\,dx = \bar\rho\,|\Omega| \quad (|\Omega| \equiv \operatorname{meas}(\Omega)), \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain and $\mathbf{n}$ denotes the unit outward normal on $\partial\Omega$. System (1.1) describes the stationary motion of a compressible, viscous magnetohydrodynamic fluid. Here $\rho > 0$ denotes the density of the fluid, $\mathbf{u} \in \mathbb{R}^3$ the velocity field, $\mathbf{H} \in \mathbb{R}^3$ the magnetic field; p is the pressure, which is assumed to be a known increasing function of ρ; The viscosity coefficients μ, ζ satisfy $\mu > 0$, $\zeta = \mu + \lambda$ with $\frac{2}{3}\mu + \lambda > 0$ and $\nu > 0$ is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field; $\mathbf{f} \in \mathbb{R}^3$ is the assigned external force field; $\bar\rho > 0$ is the mean density

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of the fluid, i.e. the total mass of fluid divided by $|\Omega|$. We emphasize that the boundary condition for $\mathbf{H}$ implies that the container Ω is perfectly conducting (cf. [4]).

For Eqs. (1.1) $_{1-3}$, Zhou [23] first proved the existence of a spatially *periodic* weak solution to the steady compressible isentropic MHD equations with $p(\rho) = a\rho^{\gamma} (a > 0)$ in $\mathbb{R}^3$ for any specific heat ratio $\gamma > 1$. To the best our knowledge, this is the first result concerning the steady MHD equations of compressible flows. In this paper, we are going to establish the existence and uniqueness of a strong solution to (1.1) with boundary conditions of physical significance. For the incompressible case of (1.1), one can refer to the monograph [10] for the existence of the weak solutions. In [10], Gerbeau et al. also considered several kinds of unsteady problems and gave some numerical analysis (see also [18]).

In recent years, there have been a lot of studies on unsteady MHD by physicists and mathematicians because of its physical importance, complexity, rich phenomena, and mathematical challenges; see [2–7,11,12,14,19] and the references cited therein. In particular, the one-dimensional problem has been studied in many papers, see, for examples, [2,3,6,19] and the references cited therein. Almost all literatures mentioned above are concerned with the Cauchy problem for (1.1) $_{1-3}$ or the initial–boundary problem for compressible MHD equations, where the boundary condition for magnetic field $\mathbf{H}$ is homogeneous Dirichlet boundary. In contrast with the extensive researches on unsteady MHD flow, we find that there are only few results concerning the steady flow.

The aim of this paper is to present a simple way to show the existence and uniqueness of a strong solution to the system (1.1), under the assumption that the external force field $\mathbf{f}$ is small in a suitable norm (see the statement of the theorem at the end of this introduction). Since we are searching for a solution in a neighborhood of the equilibrium solution $\rho = \bar{\rho}$, $\mathbf{u} = \mathbf{0}$, $\mathbf{H} = \mathbf{0}$, it is convenient to introduce the new unknown

$$\sigma := \rho - \bar{\rho}. \tag{1.2}$$

The Eqs. (1.1) thus read as

$$\begin{cases} \bar{\rho}\,\mathrm{div}\,\mathbf{u} + \mathrm{div}(\sigma\,\mathbf{u}) = 0 & \text{in } \Omega, \\ -\mu\triangle\mathbf{u} - \zeta\nabla\mathrm{div}\,\mathbf{u} + p_1\nabla\sigma = F(\sigma, \mathbf{u}, \mathbf{H}) & \text{in } \Omega, \\ \nabla\times(\nu\nabla\times\mathbf{H}) = G(\mathbf{u}, \mathbf{H}) \quad \text{and} \quad \mathrm{div}\,\mathbf{H} = 0 & \text{in } \Omega, \\ \mathbf{u}|_{\partial\Omega} = \mathbf{0}, \qquad \mathbf{H}\cdot\mathbf{n}|_{\partial\Omega} = 0, \quad \text{and} \quad (\nabla\times\mathbf{H})\times\mathbf{n}|_{\partial\Omega} = \mathbf{0} & \text{on } \partial\Omega, \\ \displaystyle\int_{\Omega} \sigma\,\mathrm{d}x = 0, \end{cases} \tag{1.3}$$

where

$$F(\sigma, \mathbf{u}, \mathbf{H}) := (\sigma + \bar{\rho})[\mathbf{f} - (\mathbf{u}\cdot\nabla)\mathbf{u}] + (\nabla\times\mathbf{H})\times\mathbf{H} + [p_1 - p'(\sigma + \bar{\rho})]\nabla\sigma \tag{1.4}$$

and

$$G(\mathbf{u}, \mathbf{H}) := \nabla\times(\mathbf{u}\times\mathbf{H}). \tag{1.5}$$

Here $p_1 = p'(\bar{\rho}) > 0$. It is clear that (1.1) and (1.3) are equivalent problems.

For the sake of simplicity, we assume $\partial\Omega \in C^4$ so we can consider as local coordinates the isothermal coordinates (cf. [20,21]). Without loss of generality, we assume $\nu = 1$. Let $H^k(\Omega)$ denote the usual Sobolev space endowed with the norm $\|\cdot\|_k$ $(k = 1, 2, 3, \ldots)$ and $\|\cdot\|_0$ denotes L^2 norm. $H^{-1}(\Omega)$ stands for the dual space to $H_0^1(\Omega)$ and is endowed with the norm $\|\cdot\|_{-1}$. Denoting by $\bar{H}^k(\Omega) = \{\sigma \in H^k(\Omega) | \int_{\Omega} \sigma\,\mathrm{d}x = 0\}$ and $W = \{\mathbf{H} \in H^1(\Omega) \mid \mathrm{div}\,\mathbf{H} = 0,\ \mathbf{H}\cdot\mathbf{n}|_{\partial\Omega} = 0\}$, we can at last state the main theorem of this paper.

Theorem 1.1. *Suppose that the pressure p is a C^2 function in a neighborhood of $\bar{\rho}$, and that $p_1 > 0$. Suppose furthermore that $\mathbf{f} \in H^1(\Omega)$ with $\|\mathbf{f}\|_1$ sufficiently small. Then the problem (1.3) admits a unique solution*

$$(\sigma, \mathbf{u}, \mathbf{H}) \in \bar{H}^2(\Omega) \times H^3(\Omega) \cap H_0^1(\Omega) \times H^2(\Omega) \cap W.$$

Moreover, there holds

$$\|\sigma\|_2 + \|\mathbf{u}\|_3 + \|\mathbf{H}\|_2 \le c\,\|\mathbf{f}\|_1,$$

where $c > 0$ is a constant depending on $\mu, \zeta, p_1, \bar{\rho}, \Omega$.

In the absence of magnetic field ($\mathbf{H} = \mathbf{0}$), as for the strong solutions of the steady Navier–Stokes equations of a compressible and viscous fluid, there are a lot of literatures. Matsumura and Nishida [15,16] first showed that there is a trivial stationary solution, when the external force $\mathbf{f}$ is given as a gradient. In [20,22], Valli and Zajaczkowski utilized a stability method to prove the existence of stationary solutions. Later Padula [17] found a solution when the ratio of viscosity coefficients λ/μ is large enough. A direct proof of the existence of stationary solutions in the general case was given by Valli [21]. Farwig [9] obtained the existence and uniqueness of a stationary solution satisfying a slip boundary condition (see also [8]). Finally, by means of a completely different method Beirão da Veiga [1] showed more general existence results in the L^p-setting. We also remark that Jiang and Zhou [13] also obtained the existence of weak solutions to the three-dimensional steady compressible Navier–Stokes equations with $\gamma > 1$ for periodic domain.