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On quasi-similarity and reducing subspaces of multiplication operator on the Fock space

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ABSTRACT

Let $F_\alpha^2(\alpha > 0)$ denote the Fock space which consists of all entire functions f in $L^2(\mathbb{C}, d\lambda_\alpha)$. We prove that the multiplication operator M_{z^n} is quasi-similar to $\bigoplus_1^n M_z$ on F_α^2. Then the reducing subspaces of M_{z^n} are characterized on F_α^2.

1. Introduction

Let $\mathbb{C}$ be the complex plane and dA be ordinary area measure on $\mathbb{C}$. Let F_α^2 $(\alpha > 0)$ denote the Fock space which consists of all entire functions f in $L^2(\mathbb{C}, d\lambda_\alpha)$, where $d\lambda_\alpha$ is the Gaussian measure, $d\lambda_\alpha(z) = \frac{\alpha}{\pi} e^{-\alpha|z|^2} dA(z)$. It is well known that F_α^2 is a closed subspace of $L^2(\mathbb{C})$, and F_α^2 is a Hilbert space with the reproducing kernel $K_\alpha(z, w) = e^{\alpha z \bar{w}}$. The normalized reproducing kernel at w is denoted by $k_w(z) = \frac{K_\alpha(z,w)}{\sqrt{K_\alpha(w,w)}} = e^{\alpha z \bar{w} - \frac{\alpha}{2}|w|^2}$. If $f, g \in F_\alpha^2$, the inner product of f and g is defined by $\langle f, g \rangle_\alpha = \int_{\mathbb{C}} f(z)\overline{g(z)} d\lambda_\alpha(z)$.

In studying an operator on a Hilbert space, it is of interest to characterize the commutant of a given operator, for such a characterization should help in understanding the structure of the operator. From the information of the commutant, people research the similar equivalence and unitary equivalence of operators. The commutant of an analytic Toeplitz operator on the Hardy space and the Bergman space has been studied extensively in the literature. We mention here that the papers [2–4,7,10,11] and the books [5,13] include a lot of the knowledge of the corresponding operator theory. In [12], Zhu obtained a complete description of the reducing subspaces of multiplication operators induced by the Blaschke product with two zeros on the Bergman space. In 2007, Jiang and Li (see [7]) proved that each analytic Toeplitz operator $M_{B(z)}$ is similar to n copies of the Bergman shift if and only if $B(z)$ is an n-Blaschke product. On the weighted Bergman space, Li (see [9]) proved that the multiplication operator M_{z^n} is similar to $\bigoplus_1^n M_z$. In 2011, Ahmadi and Hedayatian (see [1]) generalized this result to bilateral shift operators. Jiang and Zheng in [8] showed that the main result in [7] holds on the weighted Bergman space. Recently, Douglas and Kim (see [6]) studied the reducing subspaces of an analytic multiplication operator M_{z^n} on the Bergman space $A_\alpha^2(A_r)$ of the annulus A_r, and they showed that M_{z^n} has exactly 2^n reducing subspaces.

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In this paper, we first prove that the multiplication operator M_{z^n} is quasi-similar to $\bigoplus_1^n M_z$ on the Fock space. We then characterize the commutant of the operator and show that M_{z^n} has exactly 2^n reducing subspaces on the Fock space.

2. The quasi-similarity of M_{z^n} ($n \geq 2$)

Recall that $e_n(z) = \sqrt{\frac{\alpha^n}{n!}} z^n$ ($n = 0, 1, \ldots$) form an orthonormal basis of the Fock space. If we set $F_j = \mathrm{span}\{e_{nk+j}\}$ ($j = 0, 1, \ldots, n-1$), then we have the following lemma.

Lemma 2.1. *If $F_j = \mathrm{span}\{e_{nk+j}\}(j = 0, 1, \ldots, n-1)$, then*

(i) *$\{e_{nk+j}\}_{k=0}^{\infty}$ form an orthonormal basis of F_j.*
(ii) *$F_\alpha^2 = F_0 \bigoplus F_1 \bigoplus \cdots \bigoplus F_{n-1}$.*
(iii) *F_j is a reducing subspace of M_{z^n}.*

Proof. (i) Note that

$$\begin{aligned}\langle e_{nk+j}, e_{nm+j}\rangle &= \int_{\mathbb{C}} \sqrt{\frac{\alpha^{nk+j}}{(nk+j)!}} z^{nk+j} \sqrt{\frac{\alpha^{nm+j}}{(nm+j)!}} \bar{z}^{nm+j} d\lambda_\alpha(z) \\ &= \lim_{R\to+\infty} \int_{D_R} \sqrt{\frac{\alpha^{nk+j}}{(nk+j)!}} z^{nk+j} \sqrt{\frac{\alpha^{nm+j}}{(nm+j)!}} \bar{z}^{nm+j} \frac{\alpha}{\pi} e^{-\alpha|z|^2} dA(z). \end{aligned} \tag{2.1}$$

If $k = m$, we have

$$\begin{aligned}\langle e_{nk+j}, e_{nm+j}\rangle &= \lim_{R\to+\infty} \frac{\alpha^{nk+j}}{(nk+j)!} \int_0^{2\pi} d\theta \int_0^R r^{2(nk+j)} \frac{\alpha r}{\pi} e^{-\alpha r^2} dr \\ &= 1. \end{aligned} \tag{2.2}$$

If $k \neq m$, then $\langle e_{nk+j}, e_{nm+j}\rangle = 0$.

(ii) It is easy to prove that $F_j \perp F_t$, $0 \leq j \neq t \leq n-1$. Next, for $f \in F_\alpha^2$, we get that f has the form

$$f = \sum_{k=0}^{\infty} a_{0k} e_{nk} + \cdots + \sum_{k=0}^{\infty} a_{n-1,k} e_{nk+n-1}.$$

Suppose that $f = 0$. Then from

$$\left\langle \sum_{k=0}^{\infty} \sum_{j=0}^{n-1} a_{jk} e_{nk+j}, e_l \right\rangle = 0 \quad (l = 0, 1, \ldots), \tag{2.3}$$

we obtain that $a_{jk} = 0$ ($j = 0, \ldots, n-1,\ k = 0, 1, \ldots$). That is, $0 = \overbrace{0 \oplus 0 \oplus \cdots \oplus 0}^{n}$.

Therefore, $F_\alpha^2 = F_0 \bigoplus F_1 \bigoplus \cdots \bigoplus F_{n-1}$.

(iii) It is easy to see that both F_j and $F_j^\perp$ are invariant subspaces of M_{z^n}. □

Let $\{\beta(k)\}_{k=0}^{\infty}$ be a sequence of positive numbers with $\beta(0) = 1$. The Hilbert space $H^2(\beta)$ consists of all formal power series $f(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k$ ($z \in \mathbb{C}$), and $\|f\|_\beta^2 = \sum_{k=0}^{\infty} |\hat{f}(k)|^2 \beta(k)^2 < \infty$. It is known that the multiplication operator M_z is bounded on $H^2(\beta)$ if and only if $\|M_z\| = \sup_k \frac{\beta(k+1)}{\beta(k)} < \infty$. We view the Fock space F_α^2 as a Hilbert space $H^2(\beta)$, where $\beta(k) = \sqrt{\frac{k!}{\alpha^k}}$. So $\|M_z\| = \sup_k \sqrt{\frac{k+1}{\alpha}} = +\infty$.

Let H and K be complex Hilbert spaces. An operator X in $\mathcal{L}(H, K)$ is said to be quasi-invertible if X has zero kernel and dense range. Recall that for $A \in \mathcal{L}(H)$ and $B \in \mathcal{L}(K)$, A is quasi-similar to B if there exist two quasi-invertible operators S and T in $\mathcal{L}(H, K)$ and $\mathcal{L}(K, H)$ respectively such that $SA = BS$ and $AT = TB$.

Theorem 2.2. *The multiplication operator M_{z^n} is quasi-similar to $\bigoplus_1^n M_z$ on the Fock space.*

Proof. Note that

$$M_z e_k = z\sqrt{\frac{\alpha^k}{k!}} z^k = \sqrt{\frac{k+1}{\alpha}} e_{k+1}. \tag{2.4}$$