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Radon–Nikodým property and thick families of geodesics

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ABSTRACT

Banach spaces without the Radon–Nikodým property are characterized as spaces containing bilipschitz images of thick families of geodesics defined as follows. A family T of geodesics joining points u and v in a metric space is called *thick* if there is $\alpha > 0$ such that for every $g \in T$ and for any finite collection of points $r_1, \dots, r_n$ in the image of g, there is another uv-geodesic $\widetilde{g} \in T$ satisfying the conditions: $\widetilde{g}$ also passes through $r_1, \dots, r_n$, and, possibly, has some more common points with g. On the other hand, there is a finite collection of common points of g and $\widetilde{g}$ which contains $r_1, \dots, r_n$ and is such that the sum of maximal deviations of the geodesics between these common points is at least α.

1. Introduction

The Radon–Nikodým property (RNP) is one of the most important isomorphic invariants of Banach spaces. We refer to [1–3,8,13] for systematic presentations of results on the RNP, and to [6,7] for recent work on the RNP.

In the recent work on metric embeddings a substantial role is played by existence and non-existence of bilipschitz embeddings of metric spaces into Banach spaces with the RNP, see [4,5,9]. At the seminar "Nonlinear geometry of Banach spaces" (Texas A & M University, August 2009) Bill Johnson suggested the problem of metric characterization of reflexivity and the Radon–Nikodým property [14, Problem 1.1]. Some work on this problem was done in [10,12]. The purpose of this paper is to continue this work. More precisely, we are going to characterize the RNP using thick families of geodesics defined in the following way.

Definition 1.1. Let u and v be two elements in a metric space (M, d_M). A *uv-geodesic* is a distance-preserving map $g : [0, d_M(u, v)] \to M$ such that $g(0) = u$ and $g(d_M(u, v)) = v$ (where $[0, d_M(u, v)]$ is an interval of the real line with the distance inherited from $\mathbb{R}$). A family T of uv-geodesics is called *thick* if there is $\alpha > 0$ such that for every $g \in T$ and for any finite collection of points $r_1, \dots, r_n$ in the image of g, there is another uv-geodesic $\widetilde{g} \in T$ satisfying the following conditions.

- The image of $\widetilde{g}$ also contains $r_1, \dots, r_n$.
  Therefore there are $t_1, \dots, t_n \in [0, d_M(u, v)]$ such that $r_i = g(t_i) = \widetilde{g}(t_i)$.
- There are two sequences $\{q_i\}_{i=1}^m$ and $\{s_i\}_{i=1}^{m+1}$ in $[0, d_M(u, v)]$ which are listed in a non-decreasing order and satisfy the conditions:
  1. $\{q_i\}_{i=1}^m$ contains $\{t_i\}_{i=1}^n$
  2. Points $s_1, \dots, s_{m+1}$ satisfy

$$0 \le s_1 \le q_1 \le s_2 \le q_2 \le \cdots \le s_m \le q_m \le s_{m+1} \le d_M(u, v).$$

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3. $g(q_i) = \widetilde{g}(q_i)$ for all $i = 1, \ldots, m$, and

$$\sum_{i=1}^{m+1} d_M(g(s_i), \widetilde{g}(s_i)) \geq \alpha.$$

The main purpose of this paper is to prove the following result.

Theorem 1.2. *For each non-* RNP *Banach space X there exists a metric space M_X containing a thick family T_X of geodesics which admits a bilipschitz embedding into X.*

This result complements the following two results of [12].

Theorem 1.3 (*[12]*)**.** *If a Banach space X admits a bilipschitz embedding of a thick family of geodesics, then X does not have the* RNP.

Theorem 1.4 (*[12]*)**.** *For each thick family T of geodesics there exists a Banach space X which does not have the* RNP *and does not admit a bilipschitz embedding of T into X.*

Corollary 1.5 (*Of* Theorems 1.2 *and* 1.3)**.** *A Banach space X does not have the* RNP *if and only if it admits a bilipschitz embedding of some thick family of geodesics.*

Remark 1.6. Theorem 1.4 shows that the metric space M_X and the family T_X in Theorem 1.2 should depend on the space X.

We use some standard definitions of the Banach space theory and the theory of metric embeddings, see [11].

2. Proof of the main result

Proof of Theorem 1.2. We use the characterization of the RNP in terms of bushes (see [1, bottom of page 111] or [3, Theorem 2.3.6]).

Definition 2.1. Let Z be a Banach space and let $\varepsilon > 0$. A set of vectors $\{z_{n,j}\}_{n=0,j=1}^{\infty\ \ m_n}$ in Z is called an *ε-bush* if for every $n \geq 1$ there is a partition $\{A_k^n\}_{k=1}^{m_{n-1}}$ of $\{1, \ldots, m_n\}$ such that

$$\|z_{n,j} - z_{n-1,k}\| \geq \varepsilon \tag{1}$$

for every $j \in A_k^n$, and

$$z_{n-1,k} = \sum_{j \in A_k^n} \lambda_{n,j} z_{n,j} \tag{2}$$

for some $\lambda_{n,j} \geq 0$, $\sum_{j \in A_k^n} \lambda_{n,j} = 1$.

The mentioned characterization of the RNP is as follows.

Theorem 2.2 (*[1,3]*)**.** *A Banach space Z does not have the* RNP *if and only if it contains a bounded ε-bush for some $\varepsilon > 0$.*

In this theorem and below we may and shall assume that $m_0 = 1$.

It is easy to see that the direct sum of two Banach spaces with the RNP has the RNP. Because of this a subspace of codimension 1 in a non-RNP Banach space also does not have the RNP. Let $x^* \in X^*$, $\|x^*\| = 1$ be a functional which attains its norm on $x \in X$, $\|x\| = 1$. By Theorem 2.2, we can find a bounded ε-bush in $\ker x^*$. Shifting this bush by x we get a bush $\{x_{n,j}\}_{n=0,j=1}^{\infty\ \ m_n}$ satisfying the condition $x^*(x_{n,j}) = 1$ for all n and j. Consider the closure of the convex hull of the set $B_X \cup \{\pm x_{n,j}\}_{n=0,j=1}^{\infty\ \ m_n}$, where B_X is the closed unit ball of X. It is clear that this set is the unit ball of X in an equivalent norm and that in this new norm

$$\|x_{n,j}\| = 1 \quad \text{for all } n \text{ and } j. \tag{3}$$

Since the property of X which we are going to establish is clearly an isomorphic invariant, it suffices to consider the case where (3) is satisfied.

We are going to use this ε-bush to construct a thick family of geodesics in X joining 0 and $x_{0,1}$. First we construct a subset of the desired set of geodesics; this subset will be constructed as the set of limits of certain broken lines in X joining 0 and $x_{0,1}$. The constructed broken lines are also geodesics (but they do not necessarily belong to the family T_X).

The mentioned above broken lines will be constructed using representations of the form $x_{0,1} = \sum_{i=1}^m z_i$, where z_i are such that $\|x_{0,1}\| = \sum_{i=1}^m \|z_i\|$. The broken line represented by such finite sequence $z_1, \ldots, z_m$ is obtained by letting $z_0 = 0$