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Expansion formula for fractional derivatives in variational problems

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ABSTRACT

We modify the expansion formula introduced in [T.M. Atanacković, B. Stanković, An expansion formula for fractional derivatives and its applications, Fract. Calc. Appl. Anal. 7 (3) (2004) 365–378] for the left Riemann–Liouville fractional derivative in order to apply it to various problems involving fractional derivatives. As a result we obtain a new form of the fractional integration by parts formula, with the benefit of a useful approximation for the right Riemann–Liouville fractional derivative, and derive a consequence of the fractional integral inequality $\int_0^T y \cdot {}_0D_t^\alpha y\, dt \geq 0$. Further, we use this expansion formula to transform fractional optimization (minimization of a functional involving fractional derivatives) to the standard constrained optimization problem. It is shown that when the number of terms in the approximation tends to infinity, solutions to the Euler–Lagrange equations of the transformed problem converge, in a weak sense, to solutions of the original fractional Euler–Lagrange equations. An illustrative example is treated numerically.

1. Introduction

Fractional calculus has applications in many branches of physics and engineering. We mention heat conduction, nonstandard diffusion, viscoelasticity, image denoising, control, etc. (see e.g. [6,7,11,14–16,21,23]). In solving fractional differential equations that result from various applications, different procedures, both analytical and numerical ones, have been developed (cf. [10,20,23]). In the case of linear equations containing only left Riemann–Liouville derivatives the Laplace transform method is successfully used.

Recall that the left Riemann–Liouville fractional derivative of order $0 < \alpha < 1$ is defined as

$$\begin{aligned}{}_0D_t^\alpha y(t) &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{y(\tau)}{(t-\tau)^\alpha}\,d\tau \\ &= \frac{y(0)t^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{y^{(1)}(\tau)}{(t-\tau)^\alpha}\,d\tau, \quad t \in (0,T],\end{aligned}$$

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where Γ is the Euler gamma function. Here we assumed that y is an absolutely continuous function, i.e., $y \in AC([0,T])$. Similarly, the right Riemann–Liouville fractional derivative is given by

$$
\begin{aligned}
{}_tD_T^\alpha y(t) &= \frac{1}{\Gamma(1-\alpha)}\left(-\frac{d}{dt}\right)\int_t^T \frac{y(\tau)}{(\tau-t)^\alpha}\,d\tau \\
&= \frac{y(T)(T-t)^{-\alpha}}{\Gamma(1-\alpha)} - \frac{1}{\Gamma(1-\alpha)}\int_t^T \frac{y^{(1)}(\tau)}{(\tau-t)^\alpha}\,d\tau, \quad t\in[0,T).
\end{aligned}
$$

In this paper we propose a new approach in proving some well-known results for problems involving fractional derivatives. It is based on an approximation of Riemann–Liouville fractional derivatives by moments of the corresponding functions. The idea of expanding fractional derivatives in a series involving moments of a function has been introduced in [8]. Various modifications and applications of such an expansion formula can be found in the work of Pooseh, Almeida and Torres (cf. [24–26,29,30]). For instance, in [24] the authors prove an expansion formula for Riemann–Liouville fractional integrals, while in [25] an expansion formula is derived for the Hadamard fractional derivatives.

Our first goal is to establish an expansion formula for the left Riemann–Liouville fractional derivative of order $0<\alpha<1$ for a $\mathcal{C}^1$-function. The second goal is to use such a formula in certain problems with fractional derivatives. In this sense we describe the novelties of this paper, and present the problems to which the expansion formula will be applied.

First, we use the expansion formula to derive a new form of the well-known fractional integration by parts formula $\int_0^T g\,{}_0D_t^\alpha f\,dt = \int_0^T f\,{}_tD_T^\alpha g\,dt$. As a consequence, we obtain a similar approximation for the right Riemann–Liouville fractional derivative.

Second, we prove an inequality which is a consequence of the important fractional integral inequality

$$
\int_0^T y(t)\,{}_0D_t^\alpha y(t)\,dt \geq 0, \quad t\in[0,T], \tag{1}
$$

and the expansion formula. The proof of (1) and its use for solving various problems can be found in [2,12,28].

Third, we make use of the expansion formula in fractional variational problems, for obtaining approximate solutions, similarly as proposed in [4]. More precisely, we shall discuss the relation between the solutions of one-dimensional fractional variational problems and the corresponding approximated ones, prove convergence of the approximated Euler–Lagrange equations to fractional ones in a weak sense, and derive the first integrals for the approximated variational problem.

Finally we show that the expansion formula may be used to transform differential equations containing fractional derivatives to a system of integer order differential equations. This may be useful in physical applications (see Remark 2.4).

2. Expansion formula

It is known that for an analytic function y the left Riemann–Liouville derivative ${}_0D_t^\alpha y$ is expandable in a power series involving integer order derivatives as

$$
{}_0D_t^\alpha y(t) = \sum_{n=0}^{\infty}\binom{\alpha}{n}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}y^{(n)}(t), \tag{2}
$$

where $\binom{\alpha}{n} = \frac{(-1)^{n-1}\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}$ (cf. [27, p. 278]). Eq. (2) is used for an approximation of the fractional derivative if one takes only a finite number of terms in (2), i.e.,

$$
{}_0D_t^\alpha y(t) \approx \sum_{n=0}^{N}\binom{\alpha}{n}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}y^{(n)}(t).
$$

In [8,9] we proposed a new expansion formula in the following form: Let $y\in\mathcal{C}^1([0,T])$ with $y(0)=0$. Then

$$
{}_0D_t^\alpha y(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\left[V_0(y^{(1)})(t) + \sum_{p=1}^{\infty}\frac{\Gamma(p+\alpha)}{\Gamma(\alpha)p!}\frac{1}{t^p}V_p(y^{(1)})(t)\right], \tag{3}
$$

where $V_p(y)(t)$ denotes the p-th moment of the function y, $p\in\mathbb{N}$, i.e.,

$$
V_p(y)(t) = \int_0^t \tau^p y(\tau)\,d\tau, \quad p\in\mathbb{N}. \tag{4}
$$

We also remark here that the following properties of V_p hold, that can be seen directly from its definition:

$$
V_p^{(1)}(y)(t) = t^p y(t) \quad\text{and}\quad V_p(y)(0)=0, \quad \forall t\in[0,T],\ \forall p\in\mathbb{N}. \tag{5}
$$