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Asymptotic behavior of positive solutions of a nonlinear Dirichlet problem

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ABSTRACT

We take up the existence and the asymptotic behavior of a classical solution to the following semilinear Dirichlet problem

$$\begin{cases} -\Delta u = a(x)g(u), & x \in \Omega, \\ u > 0 \quad \text{in } \Omega, & u_{|\partial\Omega} = 0, \end{cases}$$

where Ω is a $C^{1,1}$-bounded domain in $\mathbb{R}^N$, $N \geq 2$ and the function a belongs to $\mathcal{C}^{\gamma}_{loc}(\Omega)$, $(0 < \gamma < 1)$ such that there exist $c_1, c_2 > 0$ satisfying for each $x \in \Omega$,

$$c_1 \delta(x)^{-\lambda_1} \exp\left(\int_{\delta(x)}^{\eta} \frac{z_1(s)}{s} ds\right) \leq a(x) \leq c_2 \delta(x)^{-\lambda_2} \exp\left(\int_{\delta(x)}^{\eta} \frac{z_2(s)}{s} ds\right),$$

where $\eta > diam(\Omega)$, $\delta(x) = dist(x, \partial\Omega)$, $\lambda_1 \leq \lambda_2 \leq 2$ and for $i \in \{1, 2\}$, z_i is a continuous function on $[0, \eta]$ with $z_i(0) = 0$.

Our arguments are based on the sub–supersolution method with Karamata regular variation theory.

1. Introduction

Let Ω be a bounded $C^{1,1}$ domain in $\mathbb{R}^N$, $N \geq 2$. We consider the following nonlinear Dirichlet problem

$$\begin{cases} -\Delta u = a(x)g(u), & x \in \Omega, \\ u > 0 \quad \text{in } \Omega, & u_{|\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where g is a nonnegative function in $C^1((0, \infty))$ and a is a positive function in $\mathcal{C}^{\gamma}_{\mathrm{loc}}(\Omega)$, satisfying some appropriate conditions related to Karamata regular variation theory.

Our main purpose is to prove the existence of a classical solution to problem (1.1) and to give estimates on such a solution.

Several articles have been devoted to the study of problems of type (1.1) involving singular or sublinear nonlinearities. We refer to [1–6,8–10,12–14,16,17,19,20] and the references therein.

Namely, the following problem

$$\begin{cases} -\Delta u = a(x)u^{\sigma}, & x \in \Omega, \ \sigma < 1, \\ u > 0 \quad \text{in } \Omega, & u_{|\partial\Omega} = 0 \end{cases} \tag{1.2}$$

is investigated by many authors under different kinds of assumptions on the weight $a(x)$ (see [1,3–6,9,10,12,16,19,20]).

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In [6], Crandall et al. considered problem (1.2) where $\sigma < -1$ and $a \equiv 1$. They proved that there exists a unique positive classical solution u of problem (1.2) satisfying

$$c_1\delta(x)^{\frac{2}{1-\sigma}} \le u(x) \le c_2\delta(x)^{\frac{2}{1-\sigma}}, \quad \text{near the boundary } \partial\Omega,$$

where c_1, c_2 are positive constants and $\delta(x) = \mathrm{dist}(x, \partial\Omega)$.

Later in [13], Lazer and McKenna extended the result stated in [6] to a large class of functions which are not necessarily bounded in $\overline{\Omega}$. Indeed they assumed that if

$$b_1\delta(x)^{-\lambda} \le a(x) \le b_2\delta(x)^{-\lambda}, \quad x \in \overline{\Omega},$$

where $b_1, b_2 > 0$ and $\lambda \in (0, 2)$, then problem (1.2) has a unique positive classical solution u in Ω such that

$$c_1\delta(x)^{\frac{2}{1-\sigma}} \le u(x) \le c_2\delta(x)^{\frac{2-\lambda}{1-\sigma}}, \quad \text{for } x \in \overline{\Omega},$$

where c_1 and c_2 are positive constants.

In [3], Brezis and Oswald studied (1.2) where $0 < \sigma < 1$ and the function a is positive and bounded in Ω and they proved that (1.2) has a unique positive solution.

Recently, applying Karamata regular variation theory, many authors have studied the asymptotic behavior of solutions of problem (1.2) (see [2–5,12,16]).

This paper is motivated by a recent result stated in [16] which will be useful to our study. To describe this result in more detail, we need some notations. We denote by $\mathcal{K}$ the set of all Karamata functions L defined on $(0, \eta]$ by

$$L(t) := c\exp\left(\int_t^\eta \frac{z(s)}{s}ds\right),$$

for some $\eta > 0$, where $c > 0$ and z is a continuous function on $[0, \eta]$ with $z(0) = 0$. It is clear that L is in $\mathcal{K}$ if and only if L is a positive function in $\mathcal{C}^1((0, \eta])$, for some $\eta > 0$, such that

$$\lim_{t\to 0^+} \frac{tL'(t)}{L(t)} = 0.$$

Let $d := \mathrm{diam}(\Omega)$ and $\eta > d$. For $\lambda \le 2$, $\sigma < 1$ and $L \in \mathcal{K}$ such that $\int_0^\eta t^{1-\lambda}L(t)dt < \infty$, we define the function $\Psi_{L,\lambda,\sigma}$ on $(0, d)$ by

$$\Psi_{L,\lambda,\sigma}(t) := \begin{cases} 1, & \text{if } \lambda < 1+\sigma, \\ \left(\int_t^\eta \frac{L(s)}{s}ds\right)^{\frac{1}{1-\sigma}}, & \text{if } \lambda = 1+\sigma, \\ (L(t))^{\frac{1}{1-\sigma}}, & \text{if } 1+\sigma < \lambda < 2, \\ \left(\int_0^t \frac{L(s)}{s}ds\right)^{\frac{1}{1-\sigma}}, & \text{if } \lambda = 2. \end{cases}$$

For two nonnegative functions f and g defined on a set S, the notation $f(x) \approx g(x)$, $x \in S$ means that there exists $c > 0$ such that $\frac{1}{c}f(x) \le g(x) \le cf(x)$, for all $x \in S$.

In [16], Mâagli studied problem (1.2), where $\sigma < 1$ and the function a verifies the following hypothesis.

(H$_1$) $a \in C^\gamma_{\mathrm{loc}}(\Omega)$, $(0 < \gamma < 1)$ satisfying for each $x \in \Omega$,

$$a(x) \approx \delta(x)^{-\lambda}L(\delta(x)),$$

where $\lambda \le 2$ and $L \in \mathcal{K}$ such that $\int_0^\eta t^{1-\lambda}L(t)dt < \infty$.

Then, by using the sub–supersolution method and some potential theory tools, the author proved in [16] the following result.

Theorem 1. *Assume* (H$_1$). *Then for each* $\sigma < 1$, *problem* (1.2) *has a unique classical solution* u *satisfying for* $x \in \Omega$,

$$u(x) \approx \delta(x)^{\min(1,\frac{2-\lambda}{1-\sigma})}\Psi_{L,\lambda,\sigma}(\delta(x)). \tag{1.3}$$

These estimates improve and generalize those established previously.

In this paper, we take up problem (1.1) and as an extension of the above result, we prove the existence of a classical solution of problem (1.1) where both terms $a(x)$ and $g(u)$ in (1.1) are required to be in a more general class of functions. Moreover, estimates (1.3) continue to hold as it can be seen in our main result. Let us consider the following hypotheses.