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Convergence rates for weighted sums in noncommutative probability space

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ABSTRACT

We study convergence rates for weighted sums of pairwise independent random variables in a noncommutative probability space of which the weights are in a von Neumann algebra. As applications, we first study convergence rates for weighted sums of random variables in the noncommutative Lorentz space, and second we study convergence rates for weighted sums of probability measures with respect to the free additive convolution.

1. Introduction

The convergence rates in the law of large numbers (LLNs) are concerned with convergence for tails of sum of random variables. In [16], Hsu and Robbins studied LLN with convergence rates and then the convergence rates were studied by several authors [12,5,4]. In particular, Baum and Katz [4] proved that for any $t > 0, r \geq 1$ and $1/2 < r/t$, the following statements are equivalent:

(i) $E\left[|X_k|^t\right] < \infty$, and $E[X_k] = \mu$ in the case of $t \geq 1$,
(ii) $\sum_{n=1}^{\infty} n^{r-2} P\left[\left|\sum_{i=1}^{n} X_i - \rho(t)n\mu\right| > n^{r/t}\epsilon\right] < \infty$ for all $\epsilon > 0$,

where $\rho = 1_{[1,\infty)}$ and $\{X_i\}$ is a sequence of independent identically distributed random variables.

In noncommutative probability theory, there are several versions of LLN, e.g., Batty [3], Jajte [17], Łuczak [21], Lindsay and Pata [20], Stoica [27], Bercovici and Pata [6,7] and the references cited therein. The convergence rates in noncommutative probability space have been established by several authors, e.g., Jajte [17], Götze and Tikhomirov [15], Chistyakov and Götze [9] and Stoica [28]. In particular, the authors in [9] gave estimates of the Lévy distance for freely independent partial sums and the author in [28] proved that Baum and Katz theorem in noncommutative Lorentz spaces.

A weighted sum of a sequence $\{X_i\}$ of random variables is of the form

$$\sum_{i=1}^{n} a_{ni} X_i, \tag{1.1}$$

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where the weighted sequence $\{a_{ni} \mid 1 \le i \le n\}$ is a triangular array. In classical probability space, the convergence rate of LLN for weighted sums of random variables has been studied by several authors [14,25,31,2,19]. The weak LLN for weighted sums of noncommutative random variables has been studied by Pata [24], Balan and Stoica [1] and Choi and Ji [10]. But the rate of convergence for weighted sums of random variables in noncommutative probability space has not yet been studied in our knowledge.

The main purpose of this paper is to establish convergence rates for weighted sums of the forms (1.1) of noncommutative random variables $\{X_i\}$ with (noncommutative) weighted sequence $\{a_{ni} \mid 1 \le i \le n\}$ in a von Neumann algebra, and then we prove convergence rates for weighted sums under certain growth and independence conditions for the weighted sequences and random variables. As applications, we study convergence rates for weighted sums of random variables in noncommutative Lorentz spaces and convergence rates for weighted additive convolution sums associated with the free independence.

This paper is organized as follows. In Section 2, we recall elementary notions in noncommutative probability theory. In Section 3, we study convergence rates for weighted sums of pairwise independent random variables in a noncommutative probability space of which the weights are in a von Neumann algebra. In Section 4, we study convergence rates for weighted sums of random variables in noncommutative Lorentz spaces. In Section 5, we establish convergence rates for weighted additive convolution sums associated with the free independence.

2. Noncommutative probability space

Let $(\mathcal{M}, \varphi)$ be a W^*-probability space (or noncommutative probability space) with a von Neumann algebra $\mathcal{M}$ (with unit $\mathbf{1}$) and a normal faithful state φ on $\mathcal{M}$. If $\mathcal{M}$ is a commutative algebra, then $(\mathcal{M}, \varphi)$ is called a classical probability space. We identify $\mathcal{M}$ by $\pi(\mathcal{M})$, where π is a faithful representation of $\mathcal{M}$ on a Hilbert space H, i.e., $\mathcal{M}$ is considered as a von Neumann subalgebra of $\mathcal{B}(H)$ the space of all bounded linear operators on H. For our convenience, throughout this paper we assume that $(\mathcal{M}, \tau)$ is a tracial W^*-probability space, i.e., $\mathcal{M}$ is von Neumann algebra and τ is a normal faithful tracial state on $\mathcal{M}$.

For $T \in \mathcal{B}(H)$, we define a closed subspace $\mathrm{Ran}(T)$ of H by

$$\mathrm{Ran}(T) = \overline{\mathrm{span}\{T(\xi) \mid \xi \in H\}}^{\|\cdot\|},$$

of which the projection from H onto $\mathrm{Ran}(T)$ is called the range projection and denoted by $R(T)$. A bounded linear operator U is called a partial isometry if for $h \in (\mathrm{Ker}\, U)^{\perp}$, $\|Uh\| = \|h\|$, where $\mathrm{Ker}(U) = \{\xi \in H \mid U(\xi) = 0\}$.

For two projections P_1, P_2 in $\mathcal{M}$, we denote $P_1 \sim P_2$ if P_1 and P_2 are equivalent, i.e., there exists a partial isometry U in $\mathcal{M}$ such that $U^*U = P_1$ and $UU^* = P_2$, and $P_1 \prec P_2$ if P_1 is equivalent to a subprojection of P_2. Note that for $T \in \mathcal{B}(H), R(T) \sim R(T^*)$ and for two projection P_1 and $P_2, R(P_1P_2) = P_1 - P_1 \wedge (\mathbf{1} - P_2)$. For any projections P_1 and P_2 in $\mathcal{M}$, we have

$$(P_1 \vee P_2 - P_2) \sim (P_1 - P_1 \wedge P_2), \tag{2.1}$$

which is well-known as the Kaplansky formula.

If $P_1 \wedge P_2 = 0$, then $P_1 \sim (P_1 \vee P_2 - P_2) \le \mathbf{1} - P_2$, which implies that

$$P_1 \prec \mathbf{1} - P_2 \quad (\text{if } P_1 \wedge P_2 = 0). \tag{2.2}$$

Then since τ is tracial, we have $\tau(P_1) = \tau(P_2)$ when $P_1 \sim P_2$, and by (2.1) we have

$$\tau(P_1 \vee P_2) \le \tau(P_1) + \tau(P_2). \tag{2.3}$$

Now, we recall the *measure topology* [23] of $\mathcal{M}$ given by the fundamental system of neighborhoods of 0: for any $\epsilon > 0$ and $\delta > 0$

$$N(\epsilon, \delta) = \{X \in \mathcal{M} \mid \text{ there exists a projection } P \in \mathcal{M} \text{ with } \tau(\mathbf{1} - P) \le \delta \text{ such that } \|XP\| \le \epsilon\}.$$

We denote by $\widetilde{\mathcal{M}}$ the completion of $\mathcal{M}$ with respect to the measure topology. Then the mappings

$$\mathcal{M} \times \mathcal{M} \ni (X, Y) \mapsto X + Y \in \mathcal{M},$$
$$\mathcal{M} \times \mathcal{M} \ni (X, Y) \mapsto XY \in \mathcal{M},$$
$$\mathcal{M} \ni X \mapsto X^* \in \mathcal{M}$$

have unique continuous extensions as mappings of $\widetilde{\mathcal{M}} \times \widetilde{\mathcal{M}} \longrightarrow \widetilde{\mathcal{M}}$, $\widetilde{\mathcal{M}} \times \widetilde{\mathcal{M}} \longrightarrow \widetilde{\mathcal{M}}$, $\widetilde{\mathcal{M}} \longrightarrow \widetilde{\mathcal{M}}$, respectively, with which $\widetilde{\mathcal{M}}$ becomes a topological *-algebra. For notational consistency, we denote by $L^0(\mathcal{M}, \tau)$ for $\widetilde{\mathcal{M}}$. Then we have natural inclusions:

$$\mathcal{M} \equiv L^{\infty}(\mathcal{M}, \tau) \subset L^q(\mathcal{M}, \tau) \subset L^p(\mathcal{M}, \tau) \subset \cdots \subset L^0(\mathcal{M}, \tau) = \widetilde{\mathcal{M}}$$

for $1 \le p \le q < \infty$, where $L^p(\mathcal{M}, \tau)$ is a Banach space of all elements in $L^0(\mathcal{M}, \tau)$ satisfying

$$\|X\|_p = \left[\tau\left(|X|^p\right)\right]^{1/p} \left(= \left(\int_0^{\infty} \mu_{\lambda}(X)^p d\lambda\right)^{1/p}\right) < \infty \tag{2.4}$$

where $\mu_{\lambda}(X)$ is the generalized singular number of X which is defined as (4.1) (see [26,23,13]).