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Logarithmic barrier decomposition-based interior point methods for stochastic symmetric programming

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ABSTRACT

We introduce and study two-stage stochastic symmetric programs with recourse to handle uncertainty in data defining (deterministic) symmetric programs in which a linear function is minimized over the intersection of an affine set and a symmetric cone. We present a Benders' decomposition-based interior point algorithm for solving these problems and prove its polynomial complexity. Our convergence analysis proved by showing that the log barrier associated with the recourse function of stochastic symmetric programs behaves a strongly self-concordant barrier and forms a self-concordant family on the first stage solutions. Since our analysis applies to all symmetric cones, this algorithm extends Zhao's results [G. Zhao, A log barrier method with Benders' decomposition for solving two-stage stochastic linear programs, Math. Program. Ser. A 90 (2001) 507–536] for two-stage stochastic linear programs, and Mehrotra and Özevin's results [S. Mehrotra, M.G. Özevin, Decomposition-based interior point methods for two-stage stochastic semidefinite programming, SIAM J. Optim. 18 (1) (2007) 206–222] for two-stage stochastic semidefinite programs.

1. Introduction

The purpose of this paper is to introduce the two-stage stochastic symmetric programs (SSPs) with recourse and to study this problem in the dual standard form:

$$\begin{array}{ll} \max & \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} + \mathbb{E}\left[Q(\boldsymbol{x}, \omega)\right] \\ \text{s.t.} & A\boldsymbol{x} + \boldsymbol{\xi} = \boldsymbol{b} \\ & \boldsymbol{\xi} \in \mathcal{K}_1, \end{array} \tag{1}$$

where $\boldsymbol{x}$ and $\boldsymbol{\xi}$ are the first-stage decision variables, and $Q(\boldsymbol{x}, \omega)$ is the minimum value of the problem

$$\begin{array}{ll} \max & \boldsymbol{d}(\omega)^{\mathsf{T}}\boldsymbol{y} \\ \text{s.t.} & W(\omega)\boldsymbol{y} + \boldsymbol{\zeta} = \boldsymbol{h}(\omega) - T(\omega)\boldsymbol{x} \\ & \boldsymbol{\zeta} \in \mathcal{K}_2, \end{array} \tag{2}$$

where $\boldsymbol{y}$ and $\boldsymbol{\zeta}$ are the second-stage variables, $\mathbb{E}[Q(\boldsymbol{x}, \omega)] := \int_{\Omega} Q(\boldsymbol{x}, \omega)P(d\omega)$, the matrix A and the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ are deterministic data, and the matrices $W(\omega)$ and $T(\omega)$ and the vectors $\boldsymbol{h}(\omega)$ and $\boldsymbol{d}(\omega)$ are random data whose realizations depend on an underlying outcome ω in an event space Ω with a known probability function P. The cones $\mathcal{K}_1$ and $\mathcal{K}_2$ are

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symmetric cones (i.e., closed, convex, pointed, self-dual cones with their automorphism groups acting transitively on their interiors) in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$. (Here, n_1 and n_2 are positive integers.)

Symmetric programs [19] are convex optimization problems in which we minimize a linear function over the intersection of an affine set and a symmetric cone. We shall refer to such problems, deterministic symmetric programs (DSPs), since the data defining such problems is assumed to be known with certainty. Stochastic linear programs (SLPs) with recourse (see for example [6]) have been studied since the 1950s as optimization models to handle uncertainty in data defining deterministic linear problems (DLPs). SSPs may be viewed as an extension of DSPs (by allowing uncertainty in data) on the one hand, and as an extension of SLPs (where $\mathcal{K}_1$ and $\mathcal{K}_2$ are both cones of nonnegative orthants) or, more generally, stochastic semidefinite programs (SSDPs) with recourse [4,11] (where $\mathcal{K}_1$ and $\mathcal{K}_2$ are both semidefinite cones) on the other hand.

Interior point methods [16] are considered one of the most successful class of algorithms for solving deterministic (linear and nonlinear) convex optimization problems. This gives good sense and sound reason of thinking in trying to investigate whether decomposition-based interior point algorithms are possible for stochastic programming. Zhao [21] has caught this sensible idea and has derived Benders' decomposition algorithm for SLPs based on a logarithmic barrier and proved its polynomial complexity. A Benders' decomposition based method has many advantages (see for example [21,11,7]). This method is natural for parallel computing. Mehrotra and Özevin [11] have proved important results that produced an extension of the work of Zhao [21] to the case of SSDPs and concluded by a derivation of a polynomial logarithmic barrier decomposition algorithm for this class of problems that extends Zhao's algorithm for SLPs. Mehrotra and Özevin also developed decomposition-based interior point algorithms for solving two-stage stochastic convex quadratic programs with recourse (see [13]). Concerning implementation of the decomposition algorithm, we refer the reader to another paper by Mehrotra and Özevin [12] where algorithm is implemented to solve two-stage stochastic conic programs with recourse whose underlying cones are Cartesian products of linear, second order, and semidefinite cones. Concerning algorithms for DSPs, Schmieta and Alizadeh [18,19] and Rangarajan [17] have extended interior point algorithms for DSDP to DSP. Concerning algorithms for SSPs, we know of no interior point algorithms for solving them that exploits the special structure of the symmetric cone as it is done in [18,19,17] for DSP. The question that naturally arises now is whether interior point methods could be derived for solving SSPs, and if the answer is affirmative, it is important to ask whether or not we can prove the polynomial complexity of the resulting algorithms. Our particular concern is to extend decomposition-based interior point methods from SSDPs to all SSPs based on a logarithmic barrier and to prove the polynomial complexity of the resulting algorithm by showing that the log barrier associated with the recourse function of SSPs behaves a strongly self-concordant barrier (see Nesterov and Nemirovskii [16]) and forms a self-concordant family on the first stage solutions.

Now let us indicate briefly how to solve problem (1) and (2). We assume that the event space Ω is discrete and finite. In practice, the case of interest is when the random data would have a finite number of realizations because the inputs of the modeling process require that SSPs be solved for a finite number of scenarios. The general scheme of the decomposition-based interior point algorithms is as follows. As we have an SSP with a finite scenarios, we can explicitly formulate the problem as a large scale DSP, we then use primal–dual interior point methods to solve the resulting large scale formulation directly, which can be successfully accomplished by utilizing the special structure of the underlying symmetric cones.

As mentioned earlier, the focus of this paper is on deriving log barrier decomposition algorithms for SSPs. Recently, Ariyawansa and Zhu [5] have derived a class of decomposition algorithms for SSDPs based on a volumetric barrier analogous to work of Mehrotra and Özevin [11] by utilizing the work of Anstreicher [3] for (deterministic) semidefinite programming. However, undoubtedly, the log barrier is far away the most popular and the most frequently used barrier. As the authors highlighted in [21,11], the decomposition algorithm has several attractive features and advantages in terms of its potential performance, most notably its ability to effectively speed up its execution in the early stages. The reason of that is that this algorithm does not require an explicit prior knowledge of all the scenarios and associated variables in its up front stage. Whereas optimal dual solutions of the second stage problems are used in the evaluation of the gradient and Hessian of the recourse function, in practice we can calculate the gradient and Hessian information approximately, and this explains having this desirable property. This also gives us another key advantage of the algorithm, which is its ability to be implemented in a distributed computing environment where some of the computing nodes may not be reliable.

This paper is organized as follows. In Section 2 we outline a minimal foundation of a unifying theory based on Euclidean Jordan algebras that connects all symmetric cones. This theory is very important for understanding the Jordan algebraic characterization of symmetric cones, and hence a sound understanding of the very close equivalence between the problem definition given in (1) and (2) and our constructive definition of an SSP which will be introduced in Section 3 in both the primal and dual standard forms. In Section 3 we will also see how we can use this powerful unifying theory to connect some general classes of optimization problems including SLPs, stochastic second-order cone programs (SSOCPs), and SSDPs with our problem formulation so that these general problems can be viewed as special cases of SSPs. In Section 4 we introduce a log barrier for our problem formulation. In Section 5 we show that the set of barrier functions for positive values of barrier parameter comprises a self-concordant family. Based on this property, we present short- and long-step variants of an interior point decomposition algorithm in Section 6. In Section 7 we present a convergence and complexity analysis of this class of algorithms. The last section contains some concluding remarks.

Before we end this introduction, we mention that a prototype primal interior point decomposition algorithm have been proposed recently by Chen and Mehrotra [7] for the two-stage stochastic convex optimization problem. The authors in [7] use tools from nonlinear sensitivity analysis, and develop *approximated* expressions for gradient and Hessian of the barrier function using chain-rule applied to general convex functions. Since stochastic symmetric programming is a subclass of