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Restrictions of an invertible chaotic operator to its invariant subspaces

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ABSTRACT

Let M be a closed subspace of a separable, infinite dimensional Hilbert space H with $\dim(H/M) = \infty$. We show that a bounded linear operator $A : M \to M$ has an invertible chaotic extension $T : H \to H$ if and only if A is bounded below. Motivated by our result, we further show that $A : M \to M$ has a chaotic Fredholm extension $T : H \to H$ if and only if A is left semi-Fredholm.

1. Introduction

A bounded linear operator $T : H \to H$ on a separable, infinite dimensional Hilbert space H is said to be *hypercyclic* if there is a vector h in H such that its orbit $\text{orb}(T, h) := \{h, Th, T^2h, \ldots\}$ is dense in H. Such a vector h is called a *hypercyclic vector* for T. While the orbit of a hypercyclic vector goes everywhere in the Hilbert space, there may be other vectors whose orbits are indeed finite. Such a vector is called a periodic point. More precisely, we say a vector h in H is a *periodic point* for T if $T^nh = h$ for some positive integer n depending on h. The operator T is said to be *chaotic* if T is hypercyclic and has a dense set of periodic points.

A closed subspace M of H is said to be an *invariant subspace* for T if $TM \subset M$. In the present paper, we study the prescribed behavior of a chaotic or hypercyclic operator T on its invariant subspace M. To facilitate a deeper discussion, we introduce the following definition for a bounded linear operator $A : M \to M$ on a closed subspace M of H. A bounded linear operator $T : H \to H$ is said to be a *hypercyclic extension* of A, if T is hypercyclic and $T|_M = A$. If in addition, the set of periodic points of T is dense in H, then T is said to be a *chaotic extension* of A.

Grivaux [7, Prop 1] and Chan and Turcu [4, Theorem 2] showed that if M has infinite codimension in H, then any operator $A : M \to M$ has a chaotic extension $T : H \to H$. They proved the result using different techniques but the extension T they obtained is the same. To be more precise, Grivaux [7] used the analytic function theory techniques to construct an extension T which satisfies the hypothesis of a hypercyclicity result of Godefroy and Shapiro [6]: *if* $H_+(T) = \text{span}\{\ker(T - \lambda I) : |\lambda| > 1\}$, *and* $H_-(T) = \text{span}\{\ker(T - \lambda I) : |\lambda| < 1\}$ *are dense subspaces of* H, *then* T *is hypercyclic*.

Chan and Turcu [4] used elementary Hilbert space techniques to show their extension T has an additional property that it satisfies the Hypercyclicity Criterion in the strongest form as stated in Theorem 1. However, an operator T constructed using the above result involving $H_+(T)$ and $H_-(T)$ is not guaranteed to have that property.

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As an important part of the Hypercyclicity Criterion, Chan and Turcu's result provides us with a chaotic extension T which is right invertible by its design but it must have a nontrivial kernel (see [4, p. 417]). Thus their extension T is never invertible regardless of what further assumption we make on the operator A. This naturally leads to a question whether we can construct an invertible chaotic extension $T : H \to H$, particularly when A itself is invertible. In that regard, we observe that if T has a left inverse S, then $SA = Id$ on M and hence A is bounded below. In Section 2, we show that the converse also holds true. That is, if A is bounded below on M, we construct an invertible chaotic extension T for A; see Corollary 4. To show that, we first construct a right invertible chaotic extension T for an operator A having a closed range so that $\ker T = \ker A$; see Theorem 2. Since it is important for A and T to have the same kernel, our techniques are different from those in [4,7].

Motivated by our above results on invertibility and chaotic extension, we further study in Section 3 the invertibility of the operator in the Calkin algebra, or the Fredholm operators. Specifically, we show that $A : M \to M$ has a chaotic Fredholm extension $T : H \to H$ if and only if A is left semi-Fredholm. Finally in Section 4, we show that the right invertible chaotic extension $T : H \to H$ in Theorem 2 can be constructed to have a hypercyclic subspace.

2. Invertibility of a chaotic extension

The Hypercyclicity Criterion plays a key role in our main theorem. Kitai [8] in 1982 gave the first and simplest version of the Criterion. Later in 1987, Gethner and Shapiro [5] rediscovered the Criterion in much greater generality. Since then hypercyclicity has been attracting a lot of attention as a hot research area. We exhibit in Theorem 2 a chaotic extension that satisfies the Criterion in the strongest possible sense. We now state version of the Criterion here for the sake of completeness.

Theorem 1 (*Kitai [8]; Gethner and Shapiro [5]*). *A bounded linear operator $T : H \to H$ is hypercyclic if there is a bounded linear operator $S : H \to H$ such that $TS =$ identity, and if there is a dense subset D of H such that $T^n x \to 0$, and $S^n x \to 0$ for each vector x in D.*

We are now ready to state and prove the main result of the present paper. We first remark that Chan and Turcu [4, Prop 1] showed that no bounded linear operator $A : M \to M$ on a nontrivial closed subspace M of H with $\dim(H/M) < \infty$ can have a hypercyclic extension T to H. Hence, we have to assume $\dim(H/M) = \infty$ in order to construct a hypercyclic extension T on H.

Theorem 2. *Let H be a separable, infinite dimensional Hilbert space, and M be a closed subspace of H with $\dim(H/M) = \infty$. Every bounded linear operator $A : M \to M$ with a closed range has a right invertible chaotic extension $T : H \to H$ with $\ker A = \ker T$ that satisfies the hypothesis of Theorem 1.*

Proof. First we note that $\mathrm{ran}A$ is a closed subspace of M and so M is an orthogonal sum $M = \mathrm{ran}A \oplus (\mathrm{ran}A)^\perp$. In the rest of the proof, we use the symbol $\oplus$ to denote an orthogonal sum of closed subspaces or an orthogonal sum of vectors. Since M has infinite codimension in H, we can rename M as M_0 and write H as an orthogonal sum $\bigoplus_{j\in\mathbb{Z}} M_j$ where for each $j \ge 0$, M_j is isomorphic to M, and furthermore for each $j < 0$, M_j is isomorphic to $\mathrm{ran}A$. Hence for each $j \in \mathbb{Z}$, we can write $M_j = M_j' \oplus M_j''$ where each M_j' is isomorphic to $\mathrm{ran}A$, and if $j \ge 0$ then M_j'' is isomorphic to $(\mathrm{ran}A)^\perp$, and if $j < 0$ then $M_j'' = \{0\}$. Thus, we can write H as the sum $\bigoplus_{j\in\mathbb{Z}} \left(M_j' \oplus M_j''\right)$.

Each vector h in H is written as

$$h = \left(h_j\right) = \left(\ldots, h_{-2}, h_{-1}, \underbrace{h_0}, h_1, h_2, \ldots\right),$$

where each $h_j \in M_j$ and the symbol "$\underbrace{\quad}$" indicates the zeroth component; that is, the vector in M_0. Since $M_j = M_j' \oplus M_j''$ for each $j \in \mathbb{Z}$, and indeed $M_j'' = \{0\}$ whenever $j < 0$, we can re-write h as

$$h = \left(h_j\right) = \left(\ldots, h'_{-2}, h'_{-1}, \underbrace{h_0}, h_1, h_2, \ldots\right),$$

where $h_j' \in M_j' = M_j$ for each $j < 0$.

Since $A : M \to M$ is bounded linear, the restriction map $A|_{(\ker\ A)^\perp} : (\ker\ A)^\perp \to \mathrm{ran}A$ is invertible, that is, there exists a bounded linear operator $B : \mathrm{ran}A \to (\ker A)^\perp$ such that AB is the identity on $\mathrm{ran}A$.

Let $\alpha > \max\{1, \|A\|, \|B\|\}$. By additively decomposing each vector h_j in M_j as the vector $h_j' \oplus h_j''$ in $M_j' \oplus M_j''$, and by suppressing the symbols for the isomorphisms between M_i' and M_j' for all integers i, j, and also the isomorphisms between M_i'' and M_j'' for all nonnegative integers i, j, we can define $T : H \to H$ by

$$Th = \left(\left(Th\right)_j\right) = \left(\ldots, \frac{h'_{-2}}{\alpha}, \frac{h'_{-1}}{\alpha}, \alpha h_1', \underbrace{Ah_0 + \alpha h_1}, \alpha h_2, \alpha h_3, \alpha h_4 \ldots\right).$$