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Remarks on partial regularity for suitable weak solutions of the incompressible magnetohydrodynamic equations

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ABSTRACT

In this paper, we are concerned with the partial regularity for suitable weak solutions of the tri-dimensional magnetohydrodynamic equations. With the help of the De Giorgi iteration method, we obtain the results proved by He and Xin (C. He, Z. Xin, Partial regularity of suitable weak solutions to the incompressible magnetohydrodynamic equations, J. Funct. Anal. 227 (2005) 113–152), namely, the one dimensional parabolic Hausdorff measure of the possible singular points of the velocity field and the magnetic field is zero.

1. Introduction

We consider the following incompressible Magnetohydrodynamic (MHD) equations

$$\begin{cases} v_t - \Delta v + v\cdot\nabla v - b\cdot\nabla b + \nabla\Pi = 0, & \Omega\times(0,T),\\ b_t - \Delta b + v\cdot\nabla b - b\cdot\nabla v = 0, & \Omega\times(0,T),\\ \operatorname{div} v = \operatorname{div} b = 0, & \Omega\times(0,T),\\ (v,b)|_{t=0} = (v_0,b_0) \end{cases} \tag{1.1}$$

where $\Omega\subseteq\mathbb{R}^3$ is a bounded regular domain, v, b describe the flow velocity field and the magnetic field, respectively, and the scalar function $\Pi=\pi+\frac{1}{2}b^2$ stands for the pressure. The initial data (v_0,b_0) satisfies $\operatorname{div} v_0=\operatorname{div} b_0=0$.

When the magnetic field b becomes a constant vector, MHD equations (1.1) are reduced to the 3D Navier–Stokes equations

$$v_t - \Delta v + v\cdot\nabla v + \nabla\Pi = 0, \quad \operatorname{div} v = 0. \tag{1.2}$$

The mathematical study of the Navier–Stokes equations attracts a large number of mathematicians (see [1,8,14,20] and references therein). The groundbreaking work of the 3D Navier–Stokes equations is Leray–Hopf weak solutions. Let us first recall the definition of Leray–Hopf weak solutions.

Definition 1.1. Let $v_0\in L^2(\Omega)$ with $\operatorname{div} v_0=0$. A measurable function v is said to be a Leray–Hopf weak solution with initial data v_0 to the 3D Navier–Stokes equations provided v satisfies the following properties.

(1) $v\in L^\infty([0,T];L^2(\Omega))\cap L^2([0,T];W^{1,2}(\Omega))$.
(2) v solves (1.2) in the sense of distributions.
(3) v satisfies the energy inequality

$$\|v(t)\|^2_{L^2(\Omega)} + 2\int_0^t \|\nabla v(t)\|^2_{L^2(\Omega)}ds \le \|v_0\|^2_{L^2(\Omega)}, \quad 0\le t\le T.$$

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Despite many mathematicians have studied the uniqueness and regularity of the Leray–Hopf weak solutions in $\mathbb{R}^3$ (see [4,8,16,20,21]), it is still an open problem to show the uniqueness of the Leray–Hopf weak solutions. Another kind of well-known weak solutions for the 3D Navier–Stokes is the suitable weak solutions, which is defined as follows.

Definition 1.2. A pair (v, p) is called a suitable weak solution to the Navier–Stokes equations in $\Omega \times (0, T)$, provided the following conditions are satisfied.

(1) $v \in L^\infty((0,T); L^2(\Omega)) \cap L^2((0,T); W^{1,2}(\Omega))$, $p \in L^{5/3}((0,T); L^{5/3}(\Omega))$.
(2) (v, p) solves (1.2) in $\Omega \times (0,T)$ in the sense of distributions.
(3) (v, p) satisfies the following local energy inequality in $\Omega \times (0,T)$ in the sense of distributions,

$$\partial_t\left(\frac{|v|^2}{2}\right) + \operatorname{div}\left(v\frac{|v|^2}{2}\right) + \operatorname{div}(vp) + |\nabla v|^2 - \Delta\frac{|v|^2}{2} \le 0.$$

The distinction between Leray–Hopf weak solutions and suitable weak solutions is mainly the above local energy inequality, which is the principal tool in the argument of partial regularity. The partial regularity of Navier–Stokes equations originated from Scheffer [17–19]. Caffarelli, Kohn and Nirenberg [1] proved that the one-dimensional parabolic Hausdorff measure of possible singular points of any suitable weak solution of the 3D Navier–Stokes equations is zero. Lin [15] gave a simpler proof on Caffarelli–Kohn–Nirenberg's Theorem with zero force. Later, Ladyzenskaja and Seregin [13] provided more details on [15] with nonzero force. Using De Giorgi iteration, Vasseur [22] presented a constructive proof of the partial regularity to Navier–Stokes equations.

There are many results for the Leray–Hopf weak solutions for MHD equations (1.1) similar to the Navier–Stokes equations (see [3,4,6,11]). He and Xin [12] proved the existence of suitable weak solutions for MHD equations (1.1), which satisfies the following.

(1) $v, b \in L^\infty((0,T); L^2(\Omega)) \cap L^2((0,T); W^{1,2}(\Omega))$, $p \in L^{5/3}((0,T); L^{5/3}(\Omega))$.
(2) (v, b, p) solves (1.1) in $\Omega \times (0,T)$ in the sense of distributions.
(3) (v, b, p) satisfies the following inequality in $\Omega \times (0,T)$ in the sense of distributions,

$$\begin{aligned}&\partial_t\left(\frac{|v|^2+|b|^2}{2}\right) + \operatorname{div}\left(v\left(\frac{|v|^2}{2}+\frac{|b|^2}{2}\right)\right) + \operatorname{div}(vp) + |\nabla v|^2 + |\nabla b|^2\\ &\quad - \Delta\frac{|v|^2+|b|^2}{2} + \operatorname{div}\big(b(v\cdot b)\big) \le 0.\end{aligned} \tag{1.3}$$

Furthermore, He and Xin studied the partial regularity for suitable weak solutions of MHD equations (1.1). They show that the one-dimensional parabolic Hausdorff measure of possible singular points of any suitable weak solution of the velocity field v and the magnetic field b is zero. Very recently, Han and He [10] considered the partial regularity of suitable weak solutions for incompressible magneto-hydrodynamic equations in $\mathbb{R}^4$.

The target of this paper is to establish the partial regularity for MHD equations (1.1) via the De Giorgi iteration methods used in [22]. De Giorgi [5] introduced an iteration technique to show the regularity of the elliptic equation with discontinuous coefficients. Recently, Caffarelli, Vasseur [2] used De Giorgi iteration and harmonic extension to obtain the global regularity of weak solutions for the quasi-geostrophic equation and Friedlander, Vicol [9] used De Giorgi iteration to obtain global well-posedness for an advection–diffusion equation arising in magneto-geostrophic dynamics. However, both the quasi-geostrophic equation and the advection–diffusion equation arising in magneto-geostrophic dynamics are scalar equations. In general, it is difficult to use the De Giorgi iteration method on a system. To overcome this difficulty, Vasseur found an effective split on the velocity field v of the Navier–Stokes equations to study partial regularity as follows

$$v = v\left(1 - \frac{\tilde{v}_k}{|v|}\right) + v\frac{\tilde{v}_k}{|v|}, \quad k \ge 0,$$

where $\tilde{v}_k = [|v| - (1-2^{-k})]_+$.

Compared with the Navier–Stokes equation, MHD equations are determined by the interaction of the magnetic field h and the velocity field v, which will bring more difficulty when we apply the De Giorgi iteration. It seems that one cannot directly use the above split for both v and b in Eqs. (1.1). The classical Elsasser variables [7] which are defined by $u = v + b$ and $h = v - b$ enable us to obtain the Elsasser form of the incompressible magnetohydrodynamic equations

$$\begin{cases}\partial_t u - \Delta u + h\cdot\nabla u + \nabla p = 0, & \Omega\times(0,T),\\ \partial_t h - \Delta h + u\cdot\nabla h + \nabla p = 0, & \Omega\times(0,T),\\ \operatorname{div} u = \operatorname{div} h = 0, & \Omega\times(0,T),\\ (u,h)|_{t=0} = (v_0+b_0, v_0-b_0),\end{cases} \tag{1.4}$$

where $p = \Pi$ satisfies

$$\Delta p = -\sum_{i,j=1}^{3}\partial_i\partial_j(h_j u_i). \tag{1.5}$$