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Restriction of the Fourier transform to some complex curves

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ABSTRACT

The purpose of this paper is to prove a Fourier restriction estimate for certain 2-dimensional surfaces in $\mathbb{R}^{2d}$, $d \geq 3$. These surfaces are defined by a complex curve $\gamma(z)$ of simple type, which is given by a mapping of the form

$$z \mapsto \gamma(z) = \big(z,\ z^2, \ldots,\ z^{d-1},\ \phi(z)\big)$$

where $\phi(z)$ is an analytic function on a domain $\Omega \subset \mathbb{C}$. This is regarded as a real mapping $z = (x, y) \mapsto \gamma(x, y)$ from $\Omega \subset \mathbb{R}^2$ to $\mathbb{R}^{2d}$.

Our results cover the case $\phi(z) = z^N$ for any nonnegative integer N, in all dimensions $d \geq 3$. The main result is a uniform estimate, valid when $d = 3$, where $\phi(z)$ may be taken to be an arbitrary polynomial of degree at most N. It is uniform in the sense that the operator norm is independent of the coefficients of the polynomial. These results are analogues of the uniform restricted strong type estimates in [5], valid for polynomial curves of simple type and some other classes of curves in $\mathbb{R}^d$, $d \geq 3$.

1. Introduction and statement of results

Let us consider a 'complex curve' of simple type in $\mathbb{C}^d$, $d \geq 2$. By this we mean a mapping of the following form:

$$z \mapsto \gamma(z) = \big(z, z^2, \ldots, z^{d-1}, \phi(z)\big), \quad z \in \Omega \tag{1.1}$$

where $\phi(z)$ is an analytic function on a domain $\Omega \subset \mathbb{C}$. We will regard this mapping as a 2-dimensional surface in $\mathbb{R}^{2d}$, given by the real mapping

$$z = (x, y) \mapsto \gamma(x, y) = \big(x, y, x^2 - y^2, 2xy, \ldots, \operatorname{Re}(\phi(z)), \operatorname{Im}(\phi(z))\big) \in \mathbb{R}^6.$$

In what follows we use $\mathbb{C}$ and $\mathbb{R}^2$ interchangeably whenever there is no danger of confusion.

Let us consider a Fourier restriction estimate of the following form:

$$\left(\int_{\mathbb{R}^2} |\widehat{f}(\gamma(z))|^q\, w(z)\, d\mu(z)\right)^{1/q} \leq C_p\, \|f\|_{L^p(\mathbb{R}^{2d})} \tag{1.2}$$

where $\widehat{f}(\xi)$ denotes the Fourier transform of $f \in L^p(\mathbb{R}^{2d})$, and the weight function $w(z)$ is given by

$$w(z) = |\tau(z)|^{4/(d^2+d)}, \qquad \text{where } \tau(z) = \det(\gamma'(z), \ldots, \gamma^{(d)}(z)). \tag{1.3}$$

Also, $d\mu$ denotes the surface measure given by $d\mu(z) = d\mu(\gamma(z)) = dxdy$ for $z = x + iy$. Here, $\widehat{f}(\gamma(z))$ stands for $\widehat{f}(\gamma(x, y))$.

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For γ given by (1.1), we have $\tau(z) = c_d\,\phi^{(d)}(z)$ with $c_d = 2!\cdots(d-1)!$. The expression $w(z)\,d\mu(z) = |\tau(z)|^{4/(d^2+d)}d\mu(z)$ is an analogue of the affine arclength measure for real curves (*cf.* [18,19,3]). See Section 2 for the optimality of this choice of measure.

When $d = 2$, Oberlin [23] proved the following.

Theorem 1.1 (*[23]; Theorem 4 and Example 3*). *Let $\gamma(z) = (z, \phi(z))$, where $\phi(z)$ is an analytic function on an open set $D \subset \mathbb{C}$. Suppose that $\phi'(z)$ and the map $(z_1, z_2) \mapsto (z_1 - z_2, \phi(z_1) - \phi(z_2))$ both have generic multiplicities at most N on D and D^2, respectively.*[1] *Then there is a constant $C_p(N) < \infty$ so that for all $f \in L^p(\mathbb{R}^4)$,*

$$\left(\int_D |\widehat{f}(\gamma(z))|^q\,|\phi''(z)|^{2/3}d\mu(z)\right)^{1/q} \le C_p(N)\|f\|_{L^p(\mathbb{R}^4)} \tag{1.4}$$

whenever $1/p + 1/(3q) = 1$, $1 \le p < 4/3$.

See [10] for a related result for some 2-dimensional surfaces in $\mathbb{R}^4$ which are not necessarily given by holomorphic functions, but which satisfy a certain nondegeneracy condition. (See also [17] for an analogous result for some k-dimensional surfaces in $\mathbb{R}^d$, where $d = 2k$.)

In this paper we obtain some positive results in higher dimensions. First let us assume that $\gamma(z)$ is in the form (1.1), where $\phi(z) = z^N$, $z \in \mathbb{C}$, for an integer $N \ge 0$.

Theorem 1.2. *Given integers $d \ge 3$ and $N \ge 0$, let $\gamma(z)$ be as in (1.1), with $\phi(z) = z^N$. Then there is a constant $C(N) < \infty$ so that for all $f \in L^{p_d,1}(\mathbb{R}^{2d})$,*

$$\left(\int_{\mathbb{R}^2} |\widehat{f}(\gamma(z))|^{p_d}\,w(z)\,d\mu(z)\right)^{1/p_d} \le C(N)\|f\|_{L^{p_d,1}(\mathbb{R}^{2d})} \tag{1.5}$$

where $w(z) = |\phi^{(d)}(z)|^{4/(d^2+d)}$ and $p_d = (d^2+d+2)/(d^2+d)$.

Moreover, there is a constant $C_p(N) < \infty$ such that

$$\left(\int_{\mathbb{R}^2} |\widehat{f}(\gamma(z))|^q\,w(z)\,d\mu(z)\right)^{1/q} \le C_p(N)\|f\|_{L^p(\mathbb{R}^{2d})} \tag{1.6}$$

whenever $1/p + 2/[(d^2+d)q] = 1$, $1 \le p < p_d$.

These estimates (as well as those in the next theorem) are expected to be optimal on the Lorentz scale of exponents, in view of the analogous results in the real case (see [3] and Theorems 1.4 and 1.5). However, this seems to be quite difficult to show in the present context, where the (real) dimension of the surface is $k = 2$. For instance, it is unknown if the estimate (1.15), which is dual to (1.6), fails for $q \le q_d$, $d \ge 3$, even when f is a bump function and we are in the nondegenerate case (with $w = 1$). This is related to the unsolved problem of determining the convergence exponent for the multi-dimensional Tarry's problem. In this connection, compare the statements of Theorem 1.3 (for $k = 1$) and Theorem 1.9 (for $k \ge 2$) in [1]. Notice that no information is available for the divergence of the integral in Theorem 1.9 (in [1]), while Theorem 1.3 (in [1]) gives the complete answer in the 1-dimensional case.

We show the sharpness of the condition $1/p + 2/[(d^2+d)q] = 1$ at the end of this section (see under the heading "A homogeneity argument"), and we also prove in Section 2 the optimality of the weight function $w(z)$, given after (1.5).

When $d = 3$, we get a uniform estimate valid for an *arbitrary* polynomial $\phi(z)$ of degree at most N.[2] This is an exact analogue of Theorem 1.5 for (real) curves, stated below.

Theorem 1.3. *For $d = 3$ and $N \ge 0$, let $\gamma(z) = (z, z^2, \phi(z))$, where $\phi(z)$ is an arbitrary polynomial of degree at most N. Then there is a constant $C(N) < \infty$, independent of the coefficients of $\phi(z)$, so that for all $f \in L^{7/6,1}(\mathbb{R}^6)$,*

$$\left(\int_{\mathbb{R}^2} |\widehat{f}(\gamma(z))|^{7/6}\,w(z)\,d\mu(z)\right)^{6/7} \le C(N)\|f\|_{L^{7/6,1}(\mathbb{R}^6)} \tag{1.7}$$

where $w(z) = |\phi'''(z)|^{1/3}$.

Moreover, there is a constant $C_p(N) < \infty$, independent of the coefficients of $\phi(z)$, such that

$$\left(\int_{\mathbb{R}^2} |\widehat{f}(\gamma(z))|^q\,w(z)\,d\mu(z)\right)^{1/q} \le C_p(N)\|f\|_{L^p(\mathbb{R}^6)}$$

whenever $1/p + 1/(6q) = 1$, $1 \le p < p_3 = 7/6$.

[1] Recall that $F : D \subset \mathbb{R}^k \to \mathbb{R}^k$ is said to have generic multiplicity N if $\mathrm{card}[F^{-1}(y)] \le N$ for almost all $y \in \mathbb{R}^k$. Here, $\mathrm{card}[E]$ denotes the cardinality of the set E.

[2] It will be interesting if one can show a version of Theorem 1.3 for higher dimensions ($d \ge 4$) as well as an analogue of Theorem 1.4 for complex curves.