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The interaction of waves for the ultra-relativistic Euler equations

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ABSTRACT

The interactions between nonlinear waves for the ultra-relativistic Euler equations for an ideal gas are studied. These equations are described in terms of the pressure p and the spatial part $\mathbf{u} \in \mathbb{R}^3$ of the dimensionless four-velocity. A new function, which measures the strengths of the waves of the ultra-relativistic Euler equations is presented, and sharp estimates for these strengths are derived. The interpretation of the strength for the Riemann solution is given. This function has the important implication that the strength is non increasing for the interactions of waves for the system. This study of interaction estimates also allows to determine the type of the outgoing Riemann solutions.

1. Introduction

Euler's equations (relativistic or classic) deal with an ideal gas in local equilibrium, in which mean free paths and collision free times are so short that perfect isotropy is maintained about any point moving with the gas. For more details about the relativistic gas we refer to the textbook of Weinberg [11, Chapter 10], which gives a short introduction to special relativity and relativistic hydrodynamics.

In this paper we are concerned with the ultra-relativistic Euler system of conservation laws for energy and momentum, namely

$$\begin{aligned}
&(p(3+4u^2))_t + \left(4pu\sqrt{1+u^2}\right)_x = 0,\\
&\left(4pu\sqrt{1+u^2}\right)_t + (p(1+4u^2))_x = 0,
\end{aligned} \tag{1.1}$$

where $p > 0$ and $u \in \mathbb{R}$. This system of nonlinear hyperbolic conservation laws was studied by many other authors, see e.g. [1,3,5,8,10]. The ultra-relativistic Euler equations considered here turn out to show several similarities with the classical Euler equations, if we add a further decoupled equation for the particle density. But system (1.1) shows a simpler mathematical behavior than the corresponding classical Euler equations. For example, even the solution of the standard

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shock tube or the Riemann problem for the classical Euler equations of gas dynamics may lead to a vacuum region within the shock tube that complicates a rigorous mathematical analysis for the general initial value problem. In contrast, the vacuum state cannot appear in the solutions of the ultra-relativistic Euler equations with non vacuum initial data.

We are interested in the interaction estimates of nonlinear waves for the ultra-relativistic Euler system (1.1). More precisely, we consider the interaction of two shocks, of a shock and a centered rarefaction wave and of two centered rarefaction waves producing transmitted waves.

The two waves are assumed to collide with each other, where the middle state in general only disappears asymptotic in time such that a new Riemann problem is formed. Especially when rarefaction waves are involved, the interaction causes complicated transient phenomena, which are not covered by the asymptotic behavior of the interacting waves.

Before wave interaction we consider three states $(p_j, u_j) \in \mathbb{R}^+ \times \mathbb{R}, j = 1, 2, 3$. Assume that, the states (p_1, u_1) and (p_2, u_2) as well as (p_2, u_2) and (p_3, u_3) can be connected by a single lower wave and a single upper wave, respectively. There is an intermediate state (p_2, u_2) before interaction, which disappears asymptotic in time. Then, after interaction, the resulting asymptotic Riemann solution shows a new intermediate state (p_*, u_*).

One basic feature presented in this paper is based on the fact that the resulting intermediate state (p_*, u_*) is given in the explicit form

$$p_* = \frac{p_1 p_3}{p_2} \tag{1.2}$$

in the case that the incoming two waves are from different families. This is a generalization of a former result in [1], where the result was only stated for two colliding shocks from different families. However, in the remaining cases concerning the interaction of two shocks from the same family and of a shock and a centered rarefaction wave from the same family, we find that (1.2) is violated, and instead of (1.2) we give algebraic inequalities for the intermediate pressure p_* in terms of the known incoming waves.

The resulting new approach will be used to introduce a special strength function which enables us to show that the strength after interactions of single waves is non increasing. This turns out to be the main result of our paper. We do not know a similar strength function for a general 2×2 hyperbolic systems of conservation laws. In the most papers about hyperbolic systems of conservation laws a more classical approach is familiar, which uses the change of Riemann invariants as a measure of wave strength, see [12,4,8] and references therein.

The paper is organized as follows. In Sections 2 and 3 we briefly review the fundamental concepts of single shock and rarefaction wave parametrizations and the solution of the Riemann problem in [5], respectively. These are the basic tools for our analysis.

In Section 4 we introduce the new strength function, which measures the strengths of the waves of the ultra-relativistic Euler equations (1.1) in a natural way, and derive sharp estimates for these strengths in Proposition 4.1. The strength of the waves is given in explicit algebraic expressions. We also give the interpretation of the strength for the Riemann solution for system (1.1).

In Section 5 we derive the formula (1.2) for the interaction of waves from different families in Propositions 5.1 and 5.2. We study the interactions between shocks and rarefaction waves in terms of the new strength function and obtain that the strength after interactions is non increasing. The cases where the strength is conserved after interaction is given in Proposition 5.3, and the other cases of strictly decreasing strength are considered in Proposition 5.4. Finally, in Section 6 we give the conclusions and outlooks.

2. The ultra-relativistic Euler equations

In this section we need some important input from thesis [5] about the parametrization of shocks and rarefaction waves. In this section we consider the ultra-relativistic Euler system of conservation laws of energy and momentum, the so called (p, u) system:

$$\begin{aligned} &(p(3+4u^2))_t + \left(4pu\sqrt{1+u^2}\right)_x = 0, \\ &\left(4pu\sqrt{1+u^2}\right)_t + (p(1+4u^2))_x = 0, \end{aligned} \tag{2.1}$$

where $p > 0$ and $u \in \mathbb{R}$. A weak form for hyperbolic conservation laws is given in [7].

For the calculation of the eigenvalues we rewrite the 2×2 system for p and u in (2.1) in the quasilinear form

$$\begin{pmatrix} p_t \\ u_t \end{pmatrix} + \begin{pmatrix} \dfrac{2u\sqrt{1+u^2}}{3+2u^2} & \dfrac{4p}{\sqrt{1+u^2}(3+2u^2)} \\ \dfrac{3\sqrt{1+u^2}}{4p(3+2u^2)} & \dfrac{2u\sqrt{1+u^2}}{3+2u^2} \end{pmatrix} \begin{pmatrix} p_x \\ u_x \end{pmatrix} = 0. \tag{2.2}$$

A simple calculation shows that system (2.1) has characteristic velocities

$$\lambda_1 = \frac{2u\sqrt{1+u^2} - \sqrt{3}}{3+2u^2} < \lambda_3 = \frac{2u\sqrt{1+u^2} + \sqrt{3}}{3+2u^2}. \tag{2.3}$$