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Positive and sign changing solutions to a nonlinear Choquard equation ☆

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ABSTRACT

We consider the problem

$$-\Delta u + W(x)u = \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u, \quad u \in H_0^1(\Omega),$$

where Ω is an exterior domain in $\mathbb{R}^N$, $N \geq 3$, $\alpha \in (0, N)$, $p \in \left[2, \frac{2N-\alpha}{N-2}\right)$, $W \in C^0(\mathbb{R}^N)$, $\inf_{\mathbb{R}^N} W > 0$, and $W(x) \to V_\infty > 0$ as $|x| \to \infty$. Under symmetry assumptions on Ω and W, which allow finite symmetries, and some assumptions on the decay of W at infinity, we establish the existence of a positive solution and multiple sign changing solutions to this problem, having small energy.

1. Introduction

We consider the problem

$$\begin{cases} -\Delta u + (V_\infty + V(x))\, u = \left(\dfrac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u, \\ u \in H_0^1(\Omega), \end{cases} \tag{1.1}$$

where $N \geq 3$, $\alpha \in (0, N)$, $p \in \left(\frac{2N-\alpha}{N}, \frac{2N-\alpha}{N-2}\right)$ and Ω is an unbounded smooth domain in $\mathbb{R}^N$ whose complement $\mathbb{R}^N \setminus \Omega$ is bounded, possibly empty. We also assume that the potential $V_\infty + V$ satisfies

(V_0) $V \in \mathcal{C}^0(\mathbb{R}^N)$, $V_\infty \in (0, \infty)$, $\inf_{x\in\mathbb{R}^N}\{V_\infty + V(x)\} > 0$, $\lim_{|x|\to\infty} V(x) = 0$.

A special case of (1.1), relevant in physical applications, is the Choquard equation

$$-\Delta u + u = \left(\frac{1}{|x|} * |u|^2\right)u, \quad u \in H^1(\mathbb{R}^3), \tag{1.2}$$

which models an electron trapped in its own hole, and was proposed by Choquard in 1976 as an approximation to Hartree–Fock theory of a one-component plasma [13]. This equation arises in many interesting situations related to the quantum theory of large systems of nonrelativistic bosonic atoms and molecules; see for example [10,15] and the references therein. It was also proposed by Penrose in 1996 as a model for the self-gravitational collapse of a quantum mechanical wave-function [24]. In this context, problem (1.2) is usually called the nonlinear Schrödinger–Newton equation; see also [19,20].

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In 1976 Lieb [13] proved the existence and uniqueness (modulo translations) of a minimizer to problem (1.2) by using symmetric decreasing rearrangement inequalities. Later, in [16], Lions showed the existence of infinitely many radially symmetric solutions to (1.2). Further results for related problems may be found in [1,7,8,18,22,25,26] and the references therein.

In 2010, Ma and Zhao [17] considered the generalized Choquard equation

$$-\Delta u+u=\left(\frac{1}{|x|^{\alpha}}*|u|^{p}\right)|u|^{p-2}u,\quad u\in H^{1}(\mathbb{R}^{N}),\tag{1.3}$$

and proved that, for $p\geq 2$, every positive solution of it is radially symmetric and monotone decreasing about some point, under the assumption that a certain set of real numbers, defined in terms of N,α and p, is nonempty. Under the same assumption, Cingolani, Clapp and Secchi [6] recently gave some existence and multiplicity results in the electromagnetic case, and established the regularity and some decay asymptotics at infinity of the ground states of (1.3). Moroz and Van Schaftingen [21] eliminated this restriction and showed the regularity, positivity and radial symmetry of the ground states for the optimal range of parameters, and derived decay asymptotics at infinity for them, as well. These results will play an important role in our study.

In this article, we are interested in obtaining positive and sign changing solutions to problem (1.1). We study the case where both Ω and V have some symmetries. If Γ is a closed subgroup of the group $O(N)$ of linear isometries of $\mathbb{R}^N$, we denote by $\Gamma x:=\{gx:g\in\Gamma\}$ the Γ-orbit of x, by $\#\Gamma x$ its cardinality, and by

$$\ell(\Gamma):=\min\{\#\Gamma x:x\in\mathbb{R}^{N}\smallsetminus\{0\}\}.$$

We assume that Ω and V are Γ-invariant, this means that $\Gamma x\subset\Omega$ for every $x\in\Omega$ and that V is constant on Γx for each $x\in\mathbb{R}^N$. We consider a continuous group homomorphism $\phi:\Gamma\to\mathbb{Z}/2$ and we look for solutions which satisfy

$$u(gx)=\phi(g)u(x)\quad\text{for all } g\in\Gamma \text{ and } x\in\Omega.\tag{1.4}$$

A function u with this property will be called *ϕ-equivariant*. We denote by

$$G:=\ker\phi.$$

Note that, if u satisfies (1.4), then u is G-invariant. Moreover, $u(\gamma x)=-u(x)$ for every $x\in\Omega$ and $\gamma\in\phi^{-1}(-1)$. Therefore, if ϕ is an epimorphism (i.e. if it is surjective), every nontrivial solution to (1.1) which satisfies (1.4) changes sign. If $\phi\equiv 1$ is the trivial homomorphism, then $\Gamma=G$ and (1.4) simply says that u is G-invariant.

If Z is a Γ-invariant subset of $\mathbb{R}^N$ and ϕ is an epimorphism, the group $\mathbb{Z}/2$ acts on the G-orbit space $Z/G:=\{Gx:x\in Z\}$ of Z as follows: we choose $\gamma\in\Gamma$ such that $\phi(\gamma)=-1$ and we define

$$(-1)\cdot Gx:=G(\gamma x)\quad\text{for all } x\in Z.$$

This action is well defined and it does not depend on the choice of γ. We denote by

$$\Sigma:=\{x\in\mathbb{R}^{N}:|x|=1,\ \#\Gamma x=\ell(\Gamma)\},\qquad \Sigma_{0}:=\{x\in\Sigma:Gx=G(\gamma x)\}.$$

If Z is a nonempty Γ-invariant subset of $\Sigma\smallsetminus\Sigma_0$, the action of $\mathbb{Z}/2$ on its G-orbit space Z/G is free and the *Krasnoselskii genus* of Z/G, denoted genus(Z/G), is defined to be the smallest $k\in\mathbb{N}$ such that there exists a continuous map $f:Z/G\to\mathbb{S}^{k-1}:=\{x\in\mathbb{R}^k:|x|=1\}$ which is $\mathbb{Z}/2$-equivariant, i.e. $f((-1)\cdot Gz)=-f(Gz)$ for every $z\in Z$. We define genus$(\emptyset):=0$.

For each subgroup K of $O(N)$ and each K-invariant subset Z of $\mathbb{R}^N\smallsetminus\{0\}$ we set

$$\mu(Kz):=\begin{cases}\inf\{|gz-hz|:g,h\in K,\ gz\neq hz\} & \text{if } \#Kz\geq 2,\\ 2|z| & \text{if } \#Kz=1,\end{cases}$$

$$\mu_{K}(Z):=\inf_{z\in Z}\mu(Kz)\quad\text{and}\quad \mu^{K}(Z):=\sup_{z\in Z}\mu(Kz).$$

In the special case where $K=G$ and $Z=\Sigma$, we simply write

$$\mu_{G}:=\mu_{G}(\Sigma)\quad\text{and}\quad \mu^{G}:=\mu^{G}(\Sigma).$$

We only consider the case $\ell(\Gamma)<\infty$, because if all Γ-orbits of Ω are infinite it was already shown in [6, Theorem 1.1] that (1.1) has infinitely many solutions. In this case, $\mu_G>0$.

We denote by c_∞ the energy of a ground state of the problem

$$\begin{cases}-\Delta u+V_{\infty}u=\left(\dfrac{1}{|x|^{\alpha}}*|u|^{p}\right)|u|^{p-2}u,\\ u\in H^{1}(\mathbb{R}^{N}).\end{cases}$$

We shall look for solutions with small energy, i.e. which satisfy

$$\frac{p-1}{2p}\int_{\mathbb{R}^{N}}\int_{\mathbb{R}^{N}}\frac{|u(x)|^{p}|u(y)|^{p}}{|x-y|^{\alpha}}dx\,dy<\ell(\Gamma)c_{\infty}.\tag{1.5}$$

In what follows, we assume that V satisfies (V_0) and we consider two cases: the case in which V is strictly negative at infinity, and that in which V takes on nonnegative values at infinity (which includes the case $V=0$). We shall prove the following results: