



Limit cycles of polynomial differential equations with quintic homogeneous nonlinearities



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ABSTRACT

In this paper we mainly study the number of limit cycles which can bifurcate from the periodic orbits of the two centers

$$\begin{aligned}\dot{x} &= -y, & \dot{y} &= x; \\ \dot{x} &= -y(1 - (x^2 + y^2)^2), & \dot{y} &= x(1 - (x^2 + y^2)^2);\end{aligned}$$

when they are perturbed inside the class of all polynomial differential systems with quintic homogeneous nonlinearities. We do this study using the averaging theory of first, second and third orders.

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1. Introduction and statement of the results

In this paper we only consider differential equations in $\mathbb{R}^2$ of the form

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y), \quad (1)$$

where P and Q are polynomials of degree at most 5 with only homogeneous nonlinearities. We recall that a *limit cycle* of the differential equation (1) is a periodic orbit of this equation isolated in the set of all periodic orbits of Eq. (1).

The definition of limit cycles appeared in the years 1891 and 1897 in the works of Poincaré [15]. Almost immediately, in 1990, they become the main object to be studied in the statement of the second part of the 16-th Hilbert problem [9]. Later on van der Pol [16] in 1926, Liénard [11] in 1928 and Andronov [1] in 1929 shown that the periodic solution of a self-sustained oscillation of a circuit in a vacuum tube was a limit cycle in the sense defined by Poincaré. After this first observation of the existence of limit cycles in the nature, the existence, non-existence, uniqueness and other properties of the limit cycles have been intensively studied first by the mathematicians and the physicists, and more recently by the chemists, biologists, economists, etc. Nowadays the study of the limit cycles of the planar differential systems has been one of the main problems of the qualitative theory of the differential equations. See for instance the recent papers [18–20] and the references quoted there.

A *center* is a singular point of a differential system (1) for which there exists a neighborhood such that all the orbits in that neighborhood are periodic, with the exception of the singular point.

A good way of producing limit cycles is by perturbing the periodic orbits of a center. This technique has been studied intensively perturbing the periodic orbits of the centers, mainly of centers of the quadratic polynomial differential systems; see for instance the book of Christopher and Li [6], and the references therein.

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The techniques used for studying the limit cycles that can bifurcate from the periodic orbits of a center, are mainly three: Abelian integrals (see [6]), Melnikov functions (see [10]), and averaging theory (see [3]). In the plane at same order all these techniques are equivalent (see [8]), they produce the same results, but the computations can change with the different technique.

In this note we shall consider polynomial differential systems of the form

$$\dot{x} = P_1(x, y) + P_n(x, y), \quad \dot{y} = Q_1(x, y) + Q_n(x, y), \quad (2)$$

where $P_k(x, y)$ and $Q_k(x, y)$ are homogeneous polynomials of degree k , i.e. we consider polynomial differential systems with homogeneous nonlinearities. For $n = 2$ we have the class of all quadratic polynomial differential systems, whose centers have been completely classified, and there are hundreds of papers studying how many limit cycles can bifurcate from the periodic orbits of these centers; see again [6]. For the general cubic polynomial differential systems the centers are not completely classified, but for the particular class of systems (2) with $n = 3$ their centers have been classified; see [14,17]. The study of the limit cycles which can bifurcate from the periodic orbits of some centers of this last class ($n = 3$) were made in [12], and for $n = 4$, see [4]. In this work we will study the class $n = 5$.

The easiest center is the linear differential center $\dot{x} = -y, \dot{y} = x$. In fact, Iliev [10] proved that the perturbation of this center inside the class of all polynomial differential systems of degree n , using the Melnikov function at order k , produces at most $[k(n-1)/2]$ limit cycles, where $[z]$ denotes the integer part function of $z \in \mathbb{R}$. Another easy center is the degenerate center $\dot{x} = -y((x^2 + y^2)/2)^m, \dot{y} = x((x^2 + y^2)/2)^m$ with $m \geq 1$. In [2] the authors improve the bound of Iliev perturbing the mentioned degenerate center. Thus, for $n = 5$ the Iliev result is at most 2, 4 and 6 limit cycles at first, second and third orders, respectively. While if we perturb the center $\dot{x} = -y((x^2 + y^2)/2)^2, \dot{y} = x((x^2 + y^2)/2)^2$ of degree 5 we get at most 2, 4 and 7 limit cycles at first, second and third orders, respectively (see Theorem 1.1 of [2]). We remark that these results are not optimal for the following polynomial differential system

$$\begin{aligned} \dot{x} &= -y + x \sum_{s=1}^3 \varepsilon^s \lambda_s + \sum_{s=1}^3 \varepsilon^s \sum_{i=0}^5 a_{i,s} x^i y^{5-i}, \\ \dot{y} &= x + y \sum_{s=1}^3 \varepsilon^s \lambda_s + \sum_{s=1}^3 \varepsilon^s \sum_{i=0}^5 b_{i,s} x^i y^{5-i}, \end{aligned} \quad (3)$$

of degree 5, because systems (3) does not have terms of degree 2, 3 and 4, only have linear terms and homogeneous nonlinearities of degree 5.

The first objective of our work will be to provide the optimal upper bounds for the number of limit cycles which can be obtained perturbing the centers $\dot{x} = -y, \dot{y} = x$ with linear terms and homogeneous nonlinearities of degree 5 by using the averaging theory of first, second and third orders; see for more details Section 3. In other words, what is the maximum number of limit cycles of systems (3) for $\varepsilon \neq 0$ sufficiently small which bifurcate from the periodic orbits of the centers $\dot{x} = -y, \dot{y} = x$ using averaging theory of first, second and third orders, respectively? The answers to this question is given in Theorem 1.

Our second objective will be to give the optimal upper bounds for the number of limit cycles which can be obtained perturbing the center $\dot{x} = -y(1 - (x^2 + y^2)^2), \dot{y} = x(1 - (x^2 + y^2)^2)$ with linear terms and homogeneous nonlinearities of degree 5 by using the averaging theory of first, second and third orders. More precisely, what is the maximum number of limit cycles of the system

$$\begin{aligned} \dot{x} &= -y(1 - (x^2 + y^2)^2) + x \sum_{s=1}^3 \varepsilon^s \lambda_s + \sum_{s=1}^3 \varepsilon^s \sum_{i=0}^5 a_{i,s} x^i y^{5-i}, \\ \dot{y} &= x(1 - (x^2 + y^2)^2) + y \sum_{s=1}^3 \varepsilon^s \lambda_s + \sum_{s=1}^3 \varepsilon^s \sum_{i=0}^5 b_{i,s} x^i y^{5-i}, \end{aligned} \quad (4)$$

which bifurcate from the periodic orbits of the center $\dot{x} = -y(1 - (x^2 + y^2)^2), \dot{y} = x(1 - (x^2 + y^2)^2)$ using averaging theory of first, second and third orders? The answer to this question is provided in Theorem 2.

We note that $H = x^2 + y^2$ is a first integral of the two differential systems (3) and (4) when $\varepsilon = 0$. Therefore, such systems when $\varepsilon = 0$ have a center at the origin, and the periodic solutions surrounding them are circles.

Our main results are the following.

Theorem 1. For $\varepsilon \neq 0$ sufficiently small and for $k = 1, 2, 3$ the maximum number of limit cycles of system (3) which bifurcate from the periodic orbits of the center $\dot{x} = -y, \dot{y} = x$ using averaging theory of k -th order is k .

Theorem 2. For $k = 1, 2$ and 3 the maximum number of limit cycles of system (4) which bifurcate from the periodic orbits of the center $\dot{x} = -y(1 - (x^2 + y^2)^2), \dot{y} = x(1 - (x^2 + y^2)^2)$ using averaging theory of k -th order is 1, 2 and 4, respectively.

Theorems 1 and 2 are proved in Section 4.

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