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BSDEs with regime switching: Weak convergence and applications☆

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HIGHLIGHTS

- We prove weak convergence of BSDEs with regime-switching.
- The Markov chain has a two-time-scale-structure.
- Weak convergence is proved under the Meyer–Zheng topology.
- We show the convergence of the corresponding PDE system.

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ABSTRACT

This paper is concerned with a system of backward stochastic differential equations (BSDEs) with regime switching. The BSDEs are coupled by a finite-state Markov chain. The underlying Markov chain is assumed to have a two-time scale (or weak and strong interactions) structure. Namely, the states of the Markov chain can be divided into a number of groups so that the chain jumps rapidly within a group and slowly between the groups. It is shown in this paper that the original BSDE system can be approximated by a limit system in which the states in each group are aggregated out and replaced by a single state. In particular, it is proved that the solution of the original BSDE system converges weakly under the Meyer–Zheng topology as the fast jump rate goes to infinity. The limit process is a solution of aggregated BSDEs which can be determined by the corresponding martingale problem. The results are applied to a set of partial differential equations and used to validate their convergence to the corresponding limit system. Finally, a numerical example is given to demonstrate the approximation results.

1. Introduction

Backward stochastic differential equations (BSDEs) were first introduced by Paradox and Peng [15]. Since then much progresses have been made in fundamental research and applications in partial differential equations, stochastic control, and mathematical finance. In particular, the BSDE approach provides a probabilistic interpretation for semi-linear PDEs, which generalizes the celebrated Feynman–Kac formula; see [16,19] for further details. Using this interpretation, the homogenization property of semi-linear PDEs was developed in [13,14,18] and asymptotic properties of these BSDEs were obtained when the coupled SDEs converge under certain conditions. Note that, in these papers, rather than using the classical

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Skorohod topology, the Meyer–Zheng topology is used to establish the weak convergence of the BSDEs. Such topology appears to be natural under the BSDE framework. Moreover, in [17], by virtue of a "transmutation process" which is in fact a special Markov chain with constant transition intensities, the authors obtained an interpretation for a semi-linear parabolic PDE system. Furthermore, BSDEs driven by both Brownian motion and Markov chains were considered in [6,7] and BSDEs purely driven by a Markov chain (rather than a Brownian motion) were treated in [4,5], which lead to a much simpler structure. In addition to these theoretical developments, there are substantial efforts devoted to applications of BSDE with Markov chains in areas including stochastic control and finance; see, for example, [2,21].

In this paper, we consider a regime switching model consisting of switching diffusions coupled by a Markov chain. The idea of regime switching was first introduced by Hamilton [10] to describe a regime switching time series. Such model stems from the need of more realistic models that better reflect random environment. For example, the movements of a equity market can be captured following a regime switching model. The market regime could reflect the state of the underlying economy, the general mood of investors in the market, and other economic factors; see [28] for related market models and references therein.

Motivated by the above mentioned research findings and applications, we consider the forward SDE and the BSDEs driven by both Brownian motion B_t and a regime switching process α_t of the following form:

$$X_t = x + \int_0^t b(s, \alpha_s, X_s)ds + \int_0^t \sigma(s, \alpha_s, X_s)dB_s, \tag{1}$$

$$Y_t = g(X_T) + \int_t^T f(s, \alpha_s, X_s, Y_s)ds - \int_t^T Z_s dB_s - \sum_{j\in I}\int_t^T W_s(j)d\widetilde{\mathcal{V}}_s(j). \tag{2}$$

Additional conditions for these equations will be provided in the next section.

We model the switching process α_t as a continuous-time Markov chain with a finite state space. In many applications, when taking into consideration various factors, the state space of α_t is inevitably large. For example, in equity markets, the performance of an individual stock can be dependent of many factors ranging from the overall health of the economy to such details as company earnings, management, book values, etc. In this case, solving the system of equations is difficult both theoretically and computationally.

In many physical models, different elements in a large system evolve at different rates. Some of them vary rapidly and others change slowly. The dynamic system evolves as if different elements or components use different clocks or time scales. Naturally, one wants to describe the largeness and smallness in a quantitative way. For example, a period of one year in daily life is a long time and numerous events can occur in between. On the other hand, one year in the history of mankind evolution is a negligible short period. Thus, "long time" vs. "short time" and "large" vs. "small" are all in relative terms. Strictly speaking, they only make sense when appropriate comparisons are made.

To reduce the complexity involved, we use a singular perturbation approach based on a two-time-scale model. The main idea is to formulate the problem using a Markov chain with two-time-scale structure. Then the variables associated with the fast scale are "averaged out" and replaced by the corresponding stationary distributions. This gives rise to a limit problem, which is determined by the stationary distribution of the fast part as well as the slowly varying component. The solution for this limit problem provides an approximation to the original problem. In this connection, we refer to the book [25] for more illustrations and related topics.

In view of these, we can take $\alpha_t = \alpha_t^\varepsilon$ governed by the generator $Q^\varepsilon(t) = \widetilde{Q}(t)/\varepsilon + \widehat{Q}(t)$. Here $\varepsilon > 0$ is a time scale parameter and both $\widetilde{Q}(t)$ and $\widehat{Q}(t)$ are generators of a Markov chain. In this paper, $\widetilde{Q}(t)$ indicates the fast part and $\widehat{Q}(t)$ the slow part. In addition, we assume that $\widetilde{Q}(t)$ has a block-diagonal form so that each block corresponds the group of states in which α_t^ε jumps rapidly. It can be shown as in [24] that the aggregated processes of α_t^ε converges to a Markov chain. Moreover, the switching diffusion equations can also be averaged out and replaced by its limit system of equations. Such limit system can be characterized in terms of the solution of a martingale problem; see [22,26]. We refer the interesting reader to [11,23,24] and references therein for further details on the approach and applications in stochastic control and finance.

With $\alpha_t = \alpha_t^\varepsilon$, the solutions of (1) and (2) are also ε dependent, i.e., $X_t = X_t^\varepsilon$ and $Y_t = Y_t^\varepsilon$. In this paper, we focus on a probabilistic interpretation for a special kind of PDE system and the corresponding homogenization property. In particular, using the weak convergence of (1), we establish the weak convergence of corresponding BSDEs (2) under the Meyer–Zheng topology. We characterize the limit processes in terms of the corresponding martingale problem. This extends the results of [22] under the BSDE framework.

From an application point of view, we would like to point out that the framework set forth in this paper can be used to characterize a contingent claim with recursive payoff. In fact, consider a defaultable zero-coupon bond with fractional recovery in which the recovered amount depends on the pre-default value of the bond. The payoff at the maturity date is dependent of the current states of the underlying assets and the value of the claim. In view of this, using the recursive pricing method (see, for example, [1,9]) and an equivalent martingale measure, one can write the value of the contingent claim $u^\varepsilon(t, \alpha_t^\varepsilon, X_t^\varepsilon)$ as follows:

$$u^\varepsilon(t, \alpha_t^\varepsilon, X_t^\varepsilon) = E\left[g(X_T^\varepsilon) + \int_t^T f(s, \alpha_s^\varepsilon, X_s^\varepsilon, u^\varepsilon(s, \alpha_s^\varepsilon, X_s^\varepsilon))ds \middle| \mathcal{F}_t^\varepsilon \right], \tag{3}$$