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Heteroclinic solutions for the extended Fisher–Kolmogorov equation☆

Y.L. Yeun *

School of Mathematics and Systems Science, Beijing University of Aeronautics and Astronautics, 100191, Beijing, China
School of Mathematical Science, Peking University, 100875, Beijing, China

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ABSTRACT

We study the extended Fisher–Kolmogorov (EFK) equation and its variants. By variational approach, we show that, for every global minimum ξ of the potential function V, there is a pair of heteroclinic solutions, one emanating from ξ and the other terminating at ξ. We require neither superquadratic growth of V nor the presence of saddle-focus equilibria; moreover V is allowed to approach its minimal level near infinity.

1. Introduction

Fourth-order differential equations are often used to study phase transition phenomena in hydrodynamics, elasticity and nonlinear optics. For example, in materials science, they appear as Euler–Lagrange equation of the Landau–Brazovsky model, which reads

$$I(q) = \int_{-\infty}^{\infty} \left\{ \frac{\beta}{2} \left(q''\right)^2 + \frac{\lambda}{2} \left(q'\right)^2 + V(q) \right\} dt, \tag{1}$$

where $\beta, \lambda > 0$ and $V(q)$ is the nonlinear potential function. This model has recently been successfully used in describing the phase state of a copolymer and thus has received much attention; for details in this respect, please see [12] and references therein. If $V(q) = \frac{1}{4}\left(q^2 - 1\right)^2$, its Euler–Lagrange equation is recognized as the classical EFK equation.

Due to their practical importance, current research has shown a growing interest in the study of fourth-order problems arising in nonlinear elasticity and materials science. Motivated by their practical applications, we consider heteroclinic solutions of the Euler–Lagrange equation of (1) with the nonlinear potential function V satisfying A_1–A_4 specified in the next section.

To begin with, we recall the most relevant studies. The first all-round effort may be attributed to Peletier and Troy [7–10]; they gave a thorough study of the EFK equation

$$q'''' + \mu q'' - q + q^3 = 0, \tag{2}$$

where $\mu \in \mathbb{R}$. They found periodic, heteroclinic and homoclinic solutions of (2); the structural and quantitative properties of the solutions were also characterized. The main tools there involved topological shooting and variational methods. In [6],

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* Correspondence to: School of Mathematics and Systems Science, Beijing University of Aeronautics and Astronautics, 100191, Beijing, China.
E-mail address: ruanyl@math.pku.edu.cn.

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W.D. Kalies and R.C.A.M. VanderVorst considered the equation with a general potential function

$$\beta q'''' - \lambda q'' + V'(q) = 0, \quad \beta, \lambda > 0, \tag{3}$$

where $V(q)$ is supposed to be analytic, have symmetric wells with equal depth and grow superquadratically near its minima (nondegeneracy). Using the gluing method, they investigated the multibump heteroclinic and homoclinic solutions. Primary examples satisfying their assumptions are

$$V_1(q) = \frac{1}{4}\left(q^2 - 1\right)^2, \qquad V_2(q) = \frac{2}{\pi^2}\left(1 + \cos \pi q\right).$$

Continuing previous studies, W.D. Kalies, J. Kwapisz and R.C.A.M. VanderVorst [4,5] concentrated on showing the complexity of the chaotic pattern of (3) with a double-well potential and saddle-focus equilibria under mild conditions of $V(q)$; specifically, $V(q)$ grows superquadratically near infinity and has exactly two nondegenerate global minima. The presence of saddle-focus equilibria also puts additional restrictions on $V(q)$, e.g. $4\beta V_1''(\pm 1) > \lambda^2$ for $V_1(q)$. Fourth-order equations with various other potential functions are considered by [13], and in [14] the author studies a fourth-order equation with a t-dependent potential function $V(t, q)$ which is not necessarily t-periodic.

Another effort in the study of fourth-order equations was made by [2]; they studied

$$q'''' - g(q)q'' - \frac{1}{2}g'(q)\left(q'\right)^2 + V'(q) = 0, \tag{4}$$

where V possesses three minima at the same level and $g \in C^2(\mathbb{R})$. It is shown that (4) possesses a heteroclinic solution connecting consecutive equilibria or the extremal minima of V. The crucial assumptions in their arguments are: (a) $4V''(0) > g(0)^2$; (b) there are $\tilde{g} \in C^1(\mathbb{R})$ and $k < 1$ such that $g(x) > \tilde{g}(x)$ and $k\sqrt{8V(x)} \geqslant \left|\int_0^x g(z)\,dz\right|$ for all $x \in \mathbb{R}$. Basically, hypothesis (a) is a generalization of the saddle-focus requirement. If g is a constant, (4) reduces to the EFK equation, and in this case (a) is nothing but the "saddle-focus" condition. Moreover, [2] did not require nondegeneracy of the minima of $V(q)$, but this loosening of the requirements is achieved at the cost of the regularity of solutions.

In this paper, we investigate the heteroclinic solutions of the EFK equation and its variants obtained as the Euler–Lagrange equations of (1). In materials science, the potential function V exhibits complex behavior, usually has multi-wells and is not symmetric. The existing results often impose restrictive conditions on V making it less applicable to real problems. Our purpose is to relax some of these conditions. We are inspired by Rabinowitz [11], who considered the second-order Hamiltonian system

$$q'' + V'(q) = 0, \tag{5}$$

where V is periodic in q and possesses equal maxima; and also, the points where maxima of V are attained constitute a discrete set. The existence of periodic and heteroclinic solutions is shown there. The key observation of [11] is that a proper subset can be found for which the heteroclinic solutions exist. The approach of [11] is employed to study our target equations. But techniques that favor a second-order Hamiltonian system may not apply equally well to fourth-order equations. One difficulty is to identify an appropriate subset where a local minimizer resides; this is one of the crucial parts of our proof. In resolving other difficulties, we also benefit from some tricky arguments and the analysis of [7–9].

What makes our results different from the relevant results in the literature is our assumptions on the potential function V. Unlike existing results, ours do not require any symmetric properties of V or the presence of saddle-focus equilibria. Moreover, it should be emphasized that we allow V to approach its minimal level near infinity (see A_4 below), which is a weakening of restrictions in comparison to existing cases; for example, W.D.Kalies, etc. assume that V grows superquadratically at infinity, and [2] requires $\liminf_{|q|\to+\infty} V(q) > 0$ in (4). Meanwhile, we mention that vanishing potentials were previously considered for natural Lagrangian systems—see, for example, [1,3]. Our results are new and valid for a more general class of potential functions. However, it should be noted that we are interested here in the existence of heteroclinic solutions under general conditions on V, while some of the existing results, like [4,5], are trying to show the complex chaotic patterns of (3), so the purposes are slightly different.

Lastly, we remark that if $\lambda < 0$ (in (1) or (3)), the situation becomes more difficult; we will not deal with this situation in the current paper, but our approach may be updated to allow small negative λ.

2. Heteroclinic solutions

Recall that the Landau–Brazovsky model is formulated as

$$I(q) = \int_{-\infty}^{\infty} \left\{ \frac{\beta}{2}\left(q''\right)^2 + \frac{\lambda}{2}\left(q'\right)^2 + V(q) \right\} dt$$

which models the energy of block copolymer; the controlling parameters β and λ are positive. We usually write the energy on a bounded interval $[r, s]$ as

$$I_{r,s}(q) = \int_r^s \left\{ \frac{\beta}{2}\left(q''\right)^2 + \frac{\lambda}{2}\left(q'\right)^2 + V(q) \right\} dt.$$