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On a rate of convergence of truncated hypersingular integrals associated to Riesz and Bessel potentials ☆,☆☆

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ABSTRACT

The notion of a μ-smooth point of a L_p-function φ and a family of truncated hypersingular integrals depending on a parameter ε are introduced. Then the connection between the order of μ-smoothness of the function φ and the rate of convergence of the family of truncated hypersingular integrals to φ, when ε tends to 0, is obtained.

1. Introduction

The classical Riesz and Bessel potentials have been proved to be powerful tools in Harmonic Analysis and its various application. These potentials are defined in terms of the Fourier transform by

$$(I^\alpha\varphi)^\wedge(x) = |x|^{-\alpha}\varphi^\wedge(x), \quad x \in \mathbb{R}^n, 0 < \alpha < n;$$
$$(\mathfrak{J}^\alpha\varphi)^\wedge(x) = (1+|x|^2)^{-\alpha/2}\varphi^\wedge(x), \quad x \in \mathbb{R}^n, 0 < \alpha < \infty. \tag{1}$$

Here, φ is a Schwartz function which is orthogonal to all polynomials (the class of these functions is dense in $L_p(\mathbb{R}^n)$ spaces for $1 < p < \infty$, see [15]). According to the formulas (1), the potential operators I^α and $\mathfrak{J}^\alpha$, are interpreted as negative (fractional) powers of differential operators $(-\Delta) = -\sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}$ and $(E-\Delta)$, respectively. One of the important trends in potential theory is finding inversion formulas for the potential operators. Hypersingular integral theory has been appeared as a result of investigation of the subject by E. Stein [19], P. Lizorkin [9], R. Wheeden [20], M. Fisher [8], S. Samko [16,15,17], V. Nogin and S. Samko [10], B. Rubin [11–14] and many other mathematicians (see references in [15,17,13] for more detailed information). The wavelet-approach to inversion of the potentials was developed by B. Rubin [13,14], I.A. Aliev, and B. Rubin [4], and I.A. Aliev [1]; see also [5,3,2,18].

In the paper [12] (see also, [13, pp. 222–224]) B. Rubin introduced some family of "truncated" integrals $D_\varepsilon^\alpha f$ and $\mathfrak{D}_\varepsilon^\alpha f$, $(\varepsilon > 0)$, generated by the Gauss–Weierstrass semigroup, and proved that under some conditions on function $\varphi \in L_p(\mathbb{R}^n)$ and parameter $\alpha > 0$, the expressions $D_\varepsilon^\alpha I^\alpha\varphi$, and $\mathfrak{D}_\varepsilon^\alpha\mathfrak{J}^\alpha\varphi$ converge to φ as $\varepsilon \to 0^+$, pointwise (a.e.) and in the L_p-norm.

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Naturally, the following question arises: Let a function $\varphi \in L_p(\mathbb{R}^n)$ has some smoothness properties at a point $x^0 \in \mathbb{R}^n$. What connection takes place between the "order of smoothness" of a function φ and the "rate of convergence" of the family $D_\varepsilon^\alpha I^\alpha\varphi$, and $\mathfrak{D}_\varepsilon^\alpha \mathfrak{J}^\alpha\varphi$ to $\varphi(x^0)$ as $\varepsilon \to 0$? The aim of this paper is to answer this question as far as possible.

2. Auxiliary definitions and lemmas

The well-known Gauss–Weierstrass semigroup, generated by a function $f(x)$, $x \in \mathbb{R}^n$, is defined by

$$(\mathrm{U}f)(x,t) = \int_{\mathbb{R}^n} W(y,t) f(x-y)dy, \quad (t>0), \tag{2}$$

where $W(y,t)$ is the following Gauss–Weierstrass kernel:

$$W(y,t) = (4\pi t)^{-n/2} e^{-|y|^2/4t}, \quad (t>0). \tag{3}$$

For the various properties of $(\mathrm{U}f)(.,t)$, see, for instance, [12] and [13, p. 223]. The modified Gauss–Weierstrass semigroup, $\mathrm{U}_M f$, is defined as

$$(\mathrm{U}_M f)(x,t) = e^{-t}(\mathrm{U}f)(x,t), \quad (t>0, x\in\mathbb{R}^n). \tag{4}$$

For $t=0$ we set $(\mathrm{U}f)(x,0) = (\mathrm{U}_M f)(x,0) = f(x)$.

The finite difference of the function $g(t)$, $(t\in\mathbb{R})$ with order $l \in \mathbb{N}$ and step τ is defined by

$$\Delta_\tau^l[g](t) = \sum_{k=0}^{l} \binom{l}{k} (-1)^k g(t+k\tau). \tag{5}$$

By making use of $(\mathrm{U}f)(x,t)$, $(\mathrm{U}_M f)(x,t)$ and $\Delta_\tau^l[g](t)$, we introduce the following "truncated" integrals (cf. [13, p. 224 and p. 262]):

$$D_\varepsilon^\alpha f(x) = \frac{1}{\chi(\frac{\alpha}{2};l)} \int_\varepsilon^\infty \Delta_\tau^l[(\mathrm{U}f)(x,\cdot)](0)\frac{d\tau}{\tau^{1+\frac{\alpha}{2}}}; \tag{6}$$

$$\mathfrak{D}_\varepsilon^\alpha f(x) = \frac{1}{\chi(\frac{\alpha}{2};l)} \int_\varepsilon^\infty \Delta_\tau^l[(\mathrm{U}_M f)(x,\cdot)](0)\frac{d\tau}{\tau^{1+\frac{\alpha}{2}}}, \tag{7}$$

where the normalized coefficient $\chi(\alpha/2;l)$ is defined by

$$\chi(\alpha/2;l) = \int_0^\infty (1-e^{-t})^l \frac{dt}{t^{1+\alpha/2}}, \quad \left(0<\frac{\alpha}{2}<l, l\in\mathbb{N}\right).$$

As shown in the following lemma, there is a close connection between the constructions (6)–(7) and the potentials $I^\alpha\varphi$ and $\mathfrak{J}^\alpha\varphi$.

Lemma 2.1 (*Rubin [12], [13, p. 224 and p. 262]*).

(a) *Let $\varphi \in L_p(\mathbb{R}^n)$, $(1 \le p < \infty)$ and $0<\alpha<n/p$. Then for any $\varepsilon>0$ and for a.e. $x\in\mathbb{R}^n$,*

$$(D_\varepsilon^\alpha I^\alpha \varphi)(x) = \int_0^\infty K_{\alpha/2}^{(l)}(\eta)(\mathrm{U}\varphi)(x,\varepsilon\eta)\,d\eta; \tag{8}$$

(b) *Let $\varphi \in L_p(\mathbb{R}^n)$, $1 \le p \le \infty$ and $0<\alpha<\infty$. Then for any $\varepsilon>0$ and for a.e. $x\in\mathbb{R}^n$,*

$$(\mathfrak{D}_\varepsilon^\alpha \mathfrak{J}^\alpha \varphi)(x) = \int_0^\infty K_{\alpha/2}^{(l)}(\eta)(\mathrm{U}_M\varphi)(x,\varepsilon\eta)\,d\eta. \tag{9}$$

Here the function $K_{\alpha/2}^{(l)}(\eta)$ is defined as

$$K_{\alpha/2}^{(l)}(\eta) = \left(\Gamma(1+\alpha/2)\chi(\alpha/2,l)\right)^{-1}\eta^{-1}\Delta_{-1}^l[\eta_+^{\alpha/2}], \tag{10}$$

where

$$\Delta_{-1}^l[\eta_+^{\alpha/2}] = \sum_{k=0}^{l}\binom{l}{k}(-1)^k(\eta-k)_+^{\alpha/2}, \quad \textit{and} \quad a_+^{\alpha/2} = \begin{Bmatrix} a^{\alpha/2}, & \textit{if } a>0, \\ 0, & \textit{if } a\le 0, \end{Bmatrix}.$$

For the sake of convenience of the reader, we give a proof of the Lemma 2.1.