J. Math. Anal. Appl. 408 (2013) 541–546

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

On the Toeplitzness of the adjoint of composition operators[☆]

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ARTICLE INFO

Article history:
Received 11 April 2012
Available online 20 June 2013
Submitted by Thomas Ransford

Keywords:
Hardy space of the unit disk
Toeplitz operator
Composition operator

ABSTRACT

Building on techniques developed by Cowen (1988) [3] and Nazarov–Shapiro (2007) [10], it is shown that the adjoint of a composition operator, induced by a unit disk-automorphism, is not strongly asymptotically Toeplitz. This result answers Nazarov–Shapiro's question in Nazarov and Shapiro (2007) [10].

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1. Introduction

In the early 60s, Brown and Halmos [2] characterized the classical Toeplitz operators on the Hardy space H^2 of the unit disk with a simple operator equation:

The operator $T \in \mathcal{B}(H^2)$ is a Toeplitz operator if and only if $T_z^* T T_z = T$, where T_z is the unilateral forward shift.

From the matricial point of view, this fact also reveals an interesting characterization of (classical) Toeplitz operators, on H^2: T is a (classical) Toeplitz operator when its matrix, with respect to the monomial basis of H^2, has constant diagonals. Indeed, the point here, as noted by Barría and Halmos [1], is that the matrix of composing TT_z is obtained from that of T by erasing the first column, while the matrix of composing T_z^*T is obtained from that of T by erasing the first row. Hence, the matrix of $T_z^*TT_z$ is obtained from that of T by moving one step down the main diagonal, and so leaves the matrix unchanged if and only if each diagonal is constant.

Twenty years later, Barría and Halmos [1] introduced a (natural) asymptotic generalization of that operator-theoretic characterization. According to them, an operator $T \in \mathcal{B}(H^2)$ is *(strongly) asymptotically Toeplitz* if the *Toeplitz sequence of* T, given by,

$$(\mathcal{T}_n(T))_{n=0}^{\infty} := \left(T_z^{*n} T T_z^n\right)_{n=0}^{\infty}$$

converges in the strong operator topology. In 1989, A. Feintuch [7] extended their definition considering other usual topologies on $\mathcal{B}(H^2)$. We thus have three flavors of asymptotic Toeplitzness: uniform, strong and weak. More precisely, an operator $T \in \mathcal{B}(H^2)$ is called *uniformly asymptotically Toeplitz*, *strongly asymptotically Toeplitz*, and *weakly asymptotically Toeplitz*, if its Toeplitz sequence is convergent in the uniform operator topology, the strong operator topology, and the weak operator topology, respectively. For each of them the operator-limit of $(\mathcal{T}_n(T))_{n=0}^{\infty}$ is a (classical) Toeplitz operator whose symbol is called the *asymptotic symbol* of T.

[☆] This paper is based on a research which forms a part of the second author's University of Toledo Ph.D. Thesis written under Professor Željko Čučković's supervision.

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0022-247X/$ – see front matter. Published by Elsevier Inc.
http://dx.doi.org/10.1016/j.jmaa.2013.06.030

It is worth mentioning that the class of uniformly asymptotically Toeplitz operators forms a (uniformly closed) subspace of all bounded operators on H^2, and it contains both Toeplitz and compact operators. Hence, any compact perturbation of a Toeplitz operator belongs to this class of operators. But, surprisingly, Feintuch proved that these are the only uniformly asymptotically Toeplitz operators [7, Theorem 4.1]:

Theorem (*Feintuch's Characterization of Uniform Asymptotic Toeplitzness*)**.** *A bounded operator on H^2 is uniformly asymptotically Toeplitz if and only if it is a compact perturbation of a Toeplitz operator.*

Hence, if the difference of a bounded operator, on H^2, from any Toeplitz operator is not a compact operator, then it does not respect uniform asymptotical Toeplitzness. And this is one of the major tools we use to prove Theorems 4.8 and 4.10.

Recently, Nazarov and Shapiro [10], studied the Toeplitz sequence of composition operators, on H^2, in the weak, strong, and uniform operator topology, and showed that the study of such phenomena led to surprising results and interesting open problems. Among other things, they also established a weakened variant of the weak asymptotic Toeplitzness: the 'arithmetic means' of the Toeplitz sequence of a composition operator, namely,

$$\frac{1}{N+1}\sum_{n=0}^{N}\mathscr{T}_n(C_\varphi), \quad (N = 0, 1, 2, \ldots),$$

and proved that for every composition operator, except the identity, these means converge in the weak operator topology to zero [10, Theorem 2.2]. Since, among the three flavors of Toeplitzness, only strongly asymptotic Toeplitzness fails to respect adjoints [1, Example 12], they also studied the behavior of the Toeplitz sequence of the adjoint of composition operators, and proved, under each of these hypotheses:

(i) $\varphi(0) = 0$, or
(ii) $|\varphi| < 1$ a.e. on $\partial\mathbb{U}$,

on H^2, C_φ^* is strongly asymptotically Toeplitz [10, Proposition 4.1 and Theorem 4.2]. At the end of their paper [10], they stated that "We do not know any non-rotational examples of composition operators whose adjoints are not strongly asymptotically Toeplitz. Perhaps they are all!"; But, in this paper, we provide a class of composition operators whose adjoints are not strongly asymptotically Toeplitz:

Theorem. *The adjoint of composition operators, induced by non-trivial $\mathbb{U}$-automorphisms, are not strongly asymptotically Toeplitz.*

The work we describe here has its roots in [1], but, is mainly inspired by Nazarov and Shapiro [10]. Here is a brief outline of what follows. In Section 2, we set up the notation and introduce the main concepts required for what follows. Section 3 provides us with more tools and techniques to prove our result on the asymptotic Toeplitzness of adjoint of $\mathbb{U}$-automorphic composition operators.

2. Prerequisites

This introductory section is dedicated to setting up the notation and introducing the main concepts along with a collection of some fundamental facts required for what is to follow.

2.1. Notations

- The symbol $\mathbb{U}$ denotes the open unit disk of the complex plane, and $\partial\mathbb{U}$ the unit circle.
- The symbol φ always denotes a holomorphic self-mapping of $\mathbb{U}$.
- $\mathrm{Hol}(\mathbb{U})$ stands for the space of all functions holomorphic on $\mathbb{U}$.
- $\mathcal{B}(\mathcal{H})$ is the space of all bounded linear operators on some Hilbert space $\mathcal{H}$.
- the usual Lebesgue space L^2, as always, is the space of (equivalence classes of) measurable functions on $\partial\mathbb{U}$ which are square-integrable with respect to the normalized arc-length measure m ($m(\partial\mathbb{U}) = 1$).
- L^∞ denotes the (Banach) space of essentially bounded measurable functions on $\partial\mathbb{U}$, equipped with the essential supremum norm, defined as

$$\|f\|_{\text{ess}} := \inf\{C \geq 0 \mid |f(e^{\imath\theta})| \leq C \text{ for almost every } e^{\imath\theta}\}.$$

- We write H^∞ for the space of bounded holomorphic functions on $\mathbb{U}$, and denote its natural norm by $\|\cdot\|_\infty$, i.e.,

$$\|f\|_\infty := \sup_{z\in\mathbb{U}}|f(z)|, \quad (f \in H^\infty).$$

- For $f \in \mathrm{Hol}(\mathbb{U})$, we adopt the notation $\hat{f}(n)$ for the n-th coefficient in the power series expansion of f about the origin.