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Conservative solutions to a system of asymptotic variational wave equations[☆]

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ABSTRACT

This paper is concerned with an asymptotic variational wave system which models weakly nonlinear waves for a system of variational wave equations arising in the theory of nematic liquid crystals and a few other physical contexts. By constructing a global semigroup, we establish the well-posedness of the initial–boundary value problem within the class of energy-conservative solutions for initial data of finite energy.

1. Introduction

In this paper, we investigate the problem described by the system

$$u_{kt} + f(u)u_{kx} = \frac{1}{2}\int_0^x f_{u_k}\sum_{i=1}^n u_{ix}^2\,\mathrm{d}x, \quad (k = 1, 2, \ldots, n) \tag{1.1}$$

for all $t \geq 0$, $x \geq 0$, with the initial–boundary value conditions

$$u(0, x) = u_0(x), \qquad u(t, 0) = \mathbf{0}, \tag{1.2}$$

where $u = (u_1, \ldots, u_n)$ is the unknown vector function defined for $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^+$, $u_0(x) = (u_{10}, \ldots, u_{n0})(x)$, $f : \mathbb{R}^n \mapsto \mathbb{R}$ is a smooth function satisfying

$$f(\mathbf{0}) \geq 0, \qquad |\nabla f(u) - \nabla f(v)| \leq L|u - v|, \quad \forall u, v \in \mathbb{R}^n \tag{1.3}$$

for a constant L.

System (1.1) can be derived from the following variational wave equations:

$$\psi_{ktt} - c(\psi)[c(\psi)\psi_{kx}]_x = c(\psi)\sum_{i=1}^n (c_{\psi_i}\psi_{kx} - c_{\psi_k}\psi_{ix})\psi_{ix}, \quad (k = 1, 2, \ldots, n), \tag{1.4}$$

which are the Euler–Lagrange equations of a variational principle arising in the theory of nematic liquid crystals; see [1,7,8]. As in [8], we look for a weakly nonlinear asymptotic solution of (1.4) of the form

$$\psi(t, x) = \psi_0 + \varepsilon u(\varepsilon t, x - c(\psi_0)t) + O(\varepsilon^2),$$

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where $\psi = (\psi_1, \ldots, \psi_n)$, ψ_0 is a constant vector and $c(\psi_0) > 0$ is the unperturbed wave speed. Then the vector function $u(\cdot, \cdot)$ satisfies

$$\left(u_{kt} + \left(\sum_{i=1}^{n} c_{u_i}(u_0)u_i\right) u_{kx}\right)_x = \frac{1}{2}c_{u_k}(u_0)\sum_{i=1}^{n} u_{ix}^2 \quad (k = 1, 2, \ldots, n) \tag{1.5}$$

up to a scaling and reflection of the independent variables, assuming that ψ_0 is such that $\sum_{i=1}^{n} |c_{u_i}(\psi_0)| \neq 0$. Integrating system (1.5) with respect to x gives a special case of (1.1).

The case $n = 1$ for (1.5) yields the well-known Hunter–Saxton equation, which has been widely studied by many authors since it was introduced [8]. It possesses a number of nice properties [9,17] and also has an interesting geometric interpretation [11,12]. Since smooth solutions may not exist globally in time [8], it becomes necessary to consider the global existence of weak solutions. There are at least two natural distinct classes of admissible weak solutions, which are called dissipative and conservative solutions [10]. The dissipative solution loses all the energy while the conservative solution preserves its energy at the blow-up time. The existence of dissipative solutions and conservative solutions to the initial–boundary value problem of the Hunter–Saxton equation are presented among others in [2,3,13,18–21].

In [4], Bressan, Zhang and Zheng established the well-posedness of the initial–boundary value problem to the case $n = 1$ of (1.1) for initial data of finite energy. Moreover, they found that the dissipative solutions may not depend continuously on the initial data when f is non-convex.

Recently, the two-component Hunter–Saxton system

$$\begin{cases} u_{txx} + uu_{xxx} + 2u_xu_{xx} = \rho\rho_x, \\ \rho_t + (\rho u)_x = 0, \end{cases} \tag{1.6}$$

arising from the two-component Camassa–Holm equation [5,6], has attracted much attention; see for example [14–16]. We notice that system (1.6) is a particular case of (1.1) for $n = 2$.

The purpose of the present paper is to establish the global well-posedness of the problem (1.1)–(1.3) for conservative solutions. We use the method used in [4] to construct a global semigroup for conservative solutions to the problem. The uniqueness result follows directly from the constructive procedure. The global existence of dissipative solutions to system (1.1) will be addressed in a forthcoming paper.

We present the main theorem of this paper, Theorem 2.1, in Section 2. Section 3 is devoted to proving this theorem.

2. The main theorem

Before we state our main results, let us first recall the definition of solutions introduced by Bressan, Zhang and Zheng [4].

Definition 1. A vector function $u(t, x)$, defined on $[0, T] \times \mathbb{R}^+$, is a solution of problem (1.1)–(1.3) if, for $k = 1, 2, \ldots, n$, the following hold.

(i) The function u_k is locally Hölder continuous with respect to both t and x. The initial and boundary conditions (1.2) hold pointwise. For each time t, the map $x \mapsto u_k(t, x)$ is absolutely continuous with $u_{kx}(t, \cdot) \in L^2_{\mathrm{loc}}(\mathbb{R}^+)$.

(ii) For any $M > 0$, the map $t \mapsto u_k(t, \cdot) \in L^2([0, M])$ is absolutely continuous and satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}u_k(t, \cdot) = -f(u)u_{kx} + \frac{1}{2}\int_0^{*} f_{u_k}(u)\sum_{i=1}^{n} u_{ix}^2 \,\mathrm{d}x \tag{2.1}$$

for a.e. $t \in [0, T]$. Here equality is understood in the sense of functions in $L^2([0, M])$.

We notice here that there is no need to consider weak solutions in the distributional sense by the local integrability assumptions $u_k(t, \cdot) \in L^2_{\mathrm{loc}}(\mathbb{R}^+)$ $(k = 1, 2, \ldots, n)$.

For each smooth solution, we can easily check that it satisfies

$$\left(\sum_{i=1}^{n} u_{ix}^2\right)_t + \left(f(u)\sum_{i=1}^{n} u_{ix}^2\right)_x = 0,$$

which implies that the existence of energy-conservative solutions is possible. We say that a solution $u = u(t, x)$ is conservative if the family of absolutely continuous measures $\{\mu_{(t)}; t \geq 0\}$ defined by $\mathrm{d}\mu_{(t)} = \sum_{i=1}^{n} u_{ix}^2(t)\mathrm{d}x$ provides a measure-valued solution to

$$\omega_t + [f(u)\omega]_x = 0, \tag{2.2}$$

that is, for every $t_2 \geq t_1 \geq 0$ and any non-negative function $\phi \in C_c^1$, there holds

$$\int \phi(t, \cdot)\,\mathrm{d}\mu_{(t)}\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} \left\{\int (\phi_t(t, \cdot) + \phi_x(t, \cdot)f(u(t, \cdot)))\,\mathrm{d}\mu_{(t)}\right\}\mathrm{d}t. \tag{2.3}$$