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Asymptotic formulas associated with psi function with applications

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ABSTRACT

We prove several asymptotic formulas associated with the psi function, and then apply them to derive the asymptotic formulas for the Euler–Mascheroni constant. Also, we give another proof of an open problem of Chen and Mortici concerning the Euler–Mascheroni constant first proved by S. Yang.

1. Introduction

The classical Euler's gamma function may be defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t.$$

Some inequalities and asymptotic formulas for the gamma function can be found (see, for example, [3,12,11,13,10]). The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x) = \Gamma'(x)/\Gamma(x)$, is known as the psi (or digamma) function. The psi function has the following asymptotic expansion (see [9, p. 32]):

$$\psi(x+t) \sim \ln x - \sum_{n=1}^{\infty} \frac{(-1)^n B_n(t)}{nx^n} \quad \text{as } x \to \infty, \tag{1.1}$$

where $B_n(t)$ stands for the Bernoulli polynomials defined by the following generating function:

$$\frac{xe^{tx}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(t)\frac{x^n}{n!}. \tag{1.2}$$

Note that the Bernoulli numbers B_n ($n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \mathbb{N} := \{1, 2, 3, \ldots\}$) are defined by $B_n := B_n(0)$ in (1.2). Setting $t = 1$ in (1.1) and noting that

$$B_n(0) = (-1)^n B_n(1) = B_n \quad \text{for } n \in \mathbb{N}_0$$

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(see [1, p. 805]), we obtain from (1.1) that

$$\psi(x+1) \sim \ln x - \sum_{n=1}^{\infty} \frac{B_n}{nx^n}$$
$$= \ln x + \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \frac{1}{240x^8} - \frac{1}{132x^{10}} + \cdots \quad \text{as } x \to \infty. \tag{1.3}$$

By using $e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}$, we deduce from (1.3) that

$$\exp(\psi(x+1)) \sim x + \frac{1}{2} + \frac{1}{24x} - \frac{1}{48x^2} + \frac{23}{5760x^3} + \frac{17}{3840x^4} - \frac{10099}{2903040x^5} - \frac{2501}{1161216x^6} + \cdots \quad \text{as } x \to \infty. \tag{1.4}$$

The main object of this paper is to give an explicit formula for determining the coefficients in the asymptotic expansion (1.4) (see Section 2), and then apply it to give another proof of an open problem of Chen and Mortici [4] concerning the Euler–Mascheroni constant first proved by S. Yang [15] (see Section 3).

2. Asymptotic expansions associated with psi function

Theorem 2.1. *The function* $\exp(\psi(x+1))$ *has the following asymptotic expansion:*

$$\exp(\psi(x+1)) \sim x\left(1 + \sum_{j=1}^{\infty} \frac{p_j}{x^j}\right) \quad \text{as } x \to \infty, \tag{2.1}$$

with the coefficients p_j (*for* $j \in \mathbb{N}$) *given by*

$$p_j = \sum_{k_1+2k_2+\cdots+jk_j=j} \frac{(-1)^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!} \left(\frac{B_1}{1}\right)^{k_1} \left(\frac{B_2}{2}\right)^{k_2} \cdots \left(\frac{B_j}{j}\right)^{k_j}, \tag{2.2}$$

where B_j *are the Bernoulli numbers, summed over all nonnegative integers* k_j *satisfying the equation*

$$k_1 + 2k_2 + \cdots + jk_j = j.$$

Proof. To determine p_j (for $j \in \mathbb{N}$) in (2.1), we first express (2.1) as follows:

$$\psi(x+1) - \ln x = \ln\left(1 + \sum_{j=1}^{m} \frac{p_j}{x^j}\right) + O(x^{-m-1}) \quad \text{as } x \to \infty. \tag{2.3}$$

By using the fundamental theorem of algebra, we see that there exist unique complex numbers $\lambda_1, \ldots, \lambda_m$ such that

$$1 + \frac{p_1}{x} + \cdots + \frac{p_m}{x^m} = \left(1 + \frac{\lambda_1}{x}\right) \cdots \left(1 + \frac{\lambda_m}{x}\right). \tag{2.4}$$

By using the following series expansion:

$$\ln\left(1 + \frac{z}{x}\right) = \sum_{j=1}^{q} \frac{(-1)^{j-1} z^j}{jx^j} + O(x^{-q-1}) \quad \text{for } |z| < |x| \text{ and } x \to \infty,$$

we obtain

$$\ln\left(1 + \frac{p_1}{x} + \cdots + \frac{p_m}{x^m}\right) = \sum_{j=1}^{m} \frac{(-1)^{j-1}\sigma_j}{jx^j} + O(x^{-m-1}) \quad \text{as } x \to \infty, \tag{2.5}$$

where

$$\sigma_j = \lambda_1^j + \cdots + \lambda_m^j \quad \text{for } j = 1, \ldots, m.$$

We then find from (1.3) and (2.5) that

$$\sigma_j = (-1)^j B_j \quad \text{for } j = 1, \ldots, m, \tag{2.6}$$