J. Math. Anal. Appl. 405 (2013) 90–103

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Existence and asymptotic stability of periodic solutions with an interior layer of reaction–advection–diffusion equations

N.N. Nefedov [a,*], L. Recke [b], K.R. Schneider [c]

[a] Moscow State University, Faculty of Physics, Department of Mathematics, Vorob'jovy Gory, 19899 Moscow, Russia
[b] Institut für Mathematik, Humboldt Universität zu Berlin, Rudower Chausse 25, 12489 Berlin, Germany
[c] Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstr. 39, 10117 Berlin, Germany

ARTICLE INFO

Article history:
Received 21 November 2012
Available online 28 March 2013
Submitted by Manuel del Pino

Keywords:
Singularly perturbed parabolic periodic problems
Exponential asymptotic stability
Krein–Rutman theorem
Lower and upper solutions

ABSTRACT

We consider a singularly perturbed parabolic periodic boundary value problem for a reaction–advection–diffusion equation. We construct the interior layer type formal asymptotics and propose a modified procedure to get asymptotic lower and upper solutions. By using sufficiently precise lower and upper solutions, we prove the existence of a periodic solution with an interior layer and estimate the accuracy of its asymptotics. Moreover, we are able to establish the asymptotic stability of this solution.

1. Statement of the problem. Construction of formal asymptotics

We consider the singularly perturbed periodic boundary value problem

$$\begin{aligned} &N_\varepsilon(u) := \varepsilon\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t}\right) - A(u,x,t)\,\frac{\partial u}{\partial x} - B(u,x,t) = 0 \\ &\quad \text{for } (x,t) \in \mathcal{D} := \{(x,t) \in \mathbb{R}^2 : -1 < x < 1, t \in \mathbb{R}\}, \\ &u(-1,t,\varepsilon) = u^{(-)}(t), \qquad u(1,t,\varepsilon) = u^{(+)}(t) \quad \text{for } t \in \mathbb{R}, \\ &u(x,t,\varepsilon) = u(x,t+T,\varepsilon) \quad \text{for } t \in \mathbb{R}, \ -1 \le x \le 1 \end{aligned} \tag{1.1}$$

for $\varepsilon \in I_{\varepsilon_0} := \{0 < \varepsilon \le \varepsilon_0\}$, $0 < \varepsilon_0 \ll 1$. The functions A, B, $u^{(-)}$ and $u^{(+)}$ are sufficiently smooth and T-periodic in t.

Our goal is to establish the existence of a solution of problem (1.1) with an interior layer with respect to x and to determine the stability of this solution. For this purpose we construct sufficiently precise asymptotic lower and upper solutions and apply the results from [5], where we developed an approach to investigate the asymptotic stability of periodic solutions to singularly perturbed reaction–advection–diffusion equations by using the theorem of Krein–Rutman. The construction of lower and upper solutions is based on the construction of a formal asymptotic approximation of the solution to (1.1) and develops further the approach used in the papers [3,4,10].

* Corresponding author.
E-mail addresses: nefedov@phys.msu.ru (N.N. Nefedov), recke@mathemtik.hu-berlin.de (L. Recke), schneider@wias-berlin.de (K.R. Schneider).

0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.03.051

1.1. Assumptions

We consider problem (1.1) under the following assumptions:

(A_0) $A, B, u^{(-)}$ and $u^{(+)}$ are sufficiently smooth and T-periodic in t.

If we put $\varepsilon = 0$ in Eq. (1.1) we get the so-called degenerate equation

$$A(u, x, t)\,\frac{\partial u}{\partial x} + B(u, x, t) = 0, \tag{1.2}$$

where t has to be considered as a parameter. Eq. (1.2) is a first order ordinary differential equation and will be studied with one of the following initial conditions from problem (1.1):

$$u(-1, t) = u^{(-)}(t) \quad \text{for } t \in \mathbb{R}, \tag{1.3}$$

$$u(1, t) = u^{(+)}(t) \quad \text{for } t \in \mathbb{R}. \tag{1.4}$$

Concerning these initial value problems we assume the following:

(A_1) The problems (1.2), (1.3) and (1.2), (1.4) have the solutions $u = \varphi^{(-)}(x, t)$ and $u = \varphi^{(+)}(x, t)$, respectively, which are defined for $(x, t) \in \overline{\mathcal{D}}$, are T-periodic in t and satisfy

$$\varphi^{(-)}(x, t) < \varphi^{(+)}(x, t) \quad \text{for } (x, t) \in \overline{\mathcal{D}},$$
$$A(\varphi^{(+)}(x, t), x, t) < 0, \qquad A(\varphi^{(-)}(x, t), x, t) > 0 \quad \text{for } (x, t) \in \overline{\mathcal{D}}.$$

To formulate the next assumptions we introduce the function $I(x, t)$ by

$$I(x, t) := \int_{\varphi^{(-)}(x,t)}^{\varphi^{(+)}(x,t)} A(u, x, t)\, du$$

and suppose

(A_2) The equation

$$I(x, t) = 0 \tag{1.5}$$

has a smooth solution $x = x_0(t)$ which is T-periodic and obeys the conditions

$$-1 < x_0(t) < 1 \quad \text{for } t \in \mathbb{R},$$
$$\int_{\varphi^{(-)}(x_0(t),t)}^{s} A(u, x_0(t), t)\, du > 0 \quad \text{for any } s \in \left(\varphi^{(-)}(x_0(t), t), \varphi^{(+)}(x_0(t), t)\right) \text{ and for } t \in \mathbb{R}.$$

(A_3) The root $x_0(t)$ of Eq. (1.5) satisfies the condition

$$\frac{\partial I}{\partial x}(x_0(t), t) < 0 \quad \text{for } t \in \mathbb{R},$$

that is, $x_0(t)$ is a simple root for all $t \in \mathbb{R}$.

Remark 1.1. Our goal is for sufficiently small ε to establish a solution to problem (1.1) that stays near $\varphi^{(-)}(x, t)$ for $x < x_0(t)$ and near $\varphi^{(+)}(x, t)$ for $x > x_0(t)$, and is a solution with an interior layer near $x_0(t)$.

Our main result is the following theorem.

Theorem 1.1. *Let the assumptions (A_0)–(A_3) be satisfied. Then, for sufficiently small ε, there exists a solution $u(x, t, \epsilon)$ of problem (1.1) such that for any small but fixed δ we have the limit relation*

$$\lim_{\epsilon \to 0} u(x, t, \epsilon) = \begin{cases} \varphi^{(-)}(x, t) & \text{for } x \in [0, x_0(t) - \delta],\ t \in \mathbb{R}, \\ \varphi^{(+)}(x, t) & \text{for } x \in [x_0(t) + \delta, 1],\ t \in \mathbb{R}, \end{cases}$$

and is asymptotically stable in the sense of Lyapunov.

We get a more precise description of the solution with an interior layer in Section 3.