J. Math. Anal. Appl. 405 (2013) 111–115

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Orbital linearization around equilibria of vector fields in $\mathbb{R}^n$ and Lie symmetries

Isaac A. García

Departament de Matemàtica, Universitat de Lleida, Avda. Jaume II, 69, 25001 Lleida, Spain

ARTICLE INFO

Article history:
Received 20 July 2012
Available online 28 March 2013
Submitted by C.E. Wayne

Keywords:
Vector field
Orbital linearization
Lie symmetry

ABSTRACT

In this paper we study some relationships between analytic vector fields $\mathcal{X}$ in $\mathbb{R}^n$ which are analytically orbitally linearizable in a neighborhood of an equilibrium point and the structure of the set $\mathfrak{N}(\mathcal{X})$ of its locally analytic normalizers (analytic infinitesimal generators of Lie point orbitally symmetries in that neighborhood). Characterization and necessary and sufficient conditions for the orbital linearization of $\mathcal{X}$ via the existence of special elements in $\mathfrak{N}(\mathcal{X})$ are given.

1. Introduction, background and main results

We consider analytic vector fields $\mathcal{X} = \sum_{i=1}^n f_i(x)\partial_{x_i}$ defined in an open set $U \subseteq \mathbb{R}^n$ containing the origin where $x = (x_1, \ldots, x_n) \in U$ and $f_i : U \to \mathbb{R}$ are analytic functions in U. We assume that $f_i(0) = 0$ for all $i = 1, \ldots, n$, i.e., the origin is an equilibrium point of $\mathcal{X}$. We also assume that the origin is a nondegenerate equilibrium in the sense that the Jacobian matrix A of the linear part of $\mathcal{X}$ at the origin $A = (\partial f_i/\partial x_j(0))$ is different from zero.

We shall use the following notation in this work: $\mathcal{X}_A$ will be the linear vector field with associated matrix $A \neq 0$. Moreover an analytic vector field $\mathcal{X}$ with linear part $\mathcal{X}_A$ is expressed as $\mathcal{X} = \mathcal{X}_A + \cdots$, where the dots denote an analytic vector field without linear terms.

A classical problem in normal form theory asks whether the foliation defined by the orbits of $\mathcal{X} = \mathcal{X}_A + \cdots$ can be analytically transformed into the foliation of an orbital linear vector field. Then $\mathcal{X}$ is called analytically *orbitally linearizable* in U if there exists a near-identity analytic change of coordinates $\Phi(x) = x + \cdots$ in U such that $\mathcal{X}$ is conjugated to a resonant vector field in the normal form $\Phi_*\mathcal{X} = f(x)\mathcal{X}_A$ with $f : U \to \mathbb{R}$ an analytic function such that $f(0) = 1$. Here Φ_* denotes the push-forward associated with the diffeomorphism Φ. In the particular case that $f(x) \equiv 1$, $\mathcal{X}$ is called analytically *linearizable* in U.

Let $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{C}$ be the spectrum of A, that is the collection of all the eigenvalues of A. Some sufficient conditions on the spectrum $\sigma(A)$ which guarantee the convergence of the normalizing transformation are known such as the condition that $\sigma(A)$ belongs to the *Poincaré domain*, i.e., the convex hull of $\sigma(A)$ in the complex plane does not contain the origin of $\mathbb{C}$.

The matrix A is said to be *resonant* if its spectrum $\sigma(A)$ satisfy the resonant condition of order m given by $\sum_{i=1}^n m_i\lambda_i = \lambda_r$ for some $r \in \{1, \ldots, n\}$ and where m_i are non-negative integers such that $m = \sum_{i=1}^n m_i \geq 2$. It is well known from Poincaré that if $\sigma(A)$ is nonresonant, then $\mathcal{X}$ can be formally linearized. Therefore, if $\sigma(A)$ belongs to the Poincaré domain and is nonresonant, then $\mathcal{X}$ is analytically linearizable.

But there are many examples of vector fields $\mathcal{X} = \mathcal{X}_A + \cdots$ where the eigenvalues of A fail to recognize if $\mathcal{X}$ is orbitally linearizable or not. It can happen for instance that all the coefficients of the nonempty set of nonlinear resonant terms in

E-mail address: garcia@matematica.udl.cat.

0022-247X/$ – see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.jmaa.2013.03.022

the normal form of $\mathcal{X}$ vanish and therefore $\mathcal{X}$ is linearizable. It is just at this point where the use of Lie symmetries of $\mathcal{X}$ is very useful to address this normal form problem.

We define the following sets. We denote by $\mathfrak{C}(\mathcal{X})$ the set of all analytic *centralizers* (or commuting vector fields $\mathcal{Z}$) of a fixed analytic vector field $\mathcal{X}$ in U, that is $\mathfrak{C}(\mathcal{X}) = \{\mathcal{Z} : [\mathcal{X}, \mathcal{Z}] = 0\}$. Here we use the standard notation $[\mathcal{X}, \mathcal{Z}] = \mathcal{X}\mathcal{Z} - \mathcal{Z}\mathcal{X}$ for the Lie bracket operation. In a similar way the set $\mathfrak{N}(\mathcal{X})$ of all analytic *normalizers* of $\mathcal{X}$ is defined as $\mathfrak{N}(\mathcal{X}) = \{\mathcal{Z} : [\mathcal{X}, \mathcal{Z}] = \Lambda\mathcal{X}\}$ for certain scalar function Λ. We call Λ the *eigenfactor* of $\mathcal{Z}$. By definition it is clear that Λ is either analytic or meromorphic with a pole at the origin, the equilibrium of $\mathcal{X}$. The notion of normalizer was introduced by Lie and Engel; see the modern translation [6]. It is obvious that $\mathfrak{C}(\mathcal{X}) \subset \mathfrak{N}(\mathcal{X})$. Moreover any $\mathcal{Z} \in \mathfrak{N}(\mathcal{X})$ is an infinitesimal generator of Lie point symmetry of $\mathcal{X}$, i.e., the flow associated with $\mathcal{Z}$ interchanges (or leaves invariant) the orbits of $\mathcal{X}$.

Here we want to emphasize that there is no algorithmic procedure to know if $\mathfrak{C}(\mathcal{X})$ is nontrivial, that is to know if $\mathfrak{C}(\mathcal{X}) \neq \mathbb{R}\,\mathcal{X}$. On the other hand clearly all the analytic sets defined previously have a formal counterpart. Thus, we define $\mathfrak{C}_{\rm for}(\mathcal{X})$ and $\mathfrak{N}_{\rm for}(\mathcal{X})$ to be the set of formal centralizers and normalizers of a given formal vector field $\mathcal{X}$, respectively. Of course, when $\mathcal{X}$ is analytic we have $\mathfrak{C}(\mathcal{X}) \subseteq \mathfrak{C}_{\rm for}(\mathcal{X})$ and $\mathfrak{N}(\mathcal{X}) \subseteq \mathfrak{N}_{\rm for}(\mathcal{X})$. We recall here that $\mathfrak{C}(\mathcal{X})$ and $\mathfrak{N}(\mathcal{X})$ are actually Lie algebras which are, in general, infinite-dimensional.

Concerning analytic normalizations using centralizers we find several results in the literature:

(i) If $\mathcal{X}$ admits an analytic normalization, then there is a nontrivial $\mathcal{Z} \in \mathfrak{C}(\mathcal{X})$ with $\mathcal{Z} \notin \mathbb{R}\mathcal{X}$; see for instance the survey [8].
(ii) In the planar case ($n = 2$) $\mathcal{X}$ admits an analytic normalization if and only if there is a nontrivial $\mathcal{Z} \in \mathfrak{C}(\mathcal{X})$ with $\mathcal{Z} \notin \mathbb{R}\mathcal{X}$; see [2].
(iii) $\mathcal{X} = \mathcal{X}_A + \cdots$ where the eigenvalues of A are nonresonant and pairwise different is analytically normalizable if and only if $\dim \mathfrak{C}(\mathcal{X}) = n$. See [1] (and also [7,4]). Of course in this case $\mathcal{X}$ is analytically linearizable due to nonresonance.
(iv) If $\hat{\mathcal{X}}$ is a normal form of $\mathcal{X}$ having $\dim \mathfrak{C}_{\rm for}(\hat{\mathcal{X}}) = k < \infty$, the eigenvalues of A satisfy Bruno's condition ω and $\dim \mathfrak{C}(\mathcal{X}) = k$, then $\mathcal{X}$ admits an analytic normalization.

Regarding analytic linearizations using centralizers, the following is known:

(i) $\mathcal{X}$ is analytically linearizable if, and only if, there exists $\mathcal{Z} \in \mathfrak{C}(\mathcal{X})$ of the form $\mathcal{Z} = \mathcal{Z}_I + \cdots$ where I is the identity matrix; see [5,3].
(ii) If $\mathcal{X}$ is analytically linearizable, then there are independent centralizers $\mathcal{Z}_i \in \mathfrak{C}(\mathcal{X})$ for $i = 1, \ldots, n$ such that $\Psi_*\mathcal{Z}_i$ are linear for some analytic normalization Ψ; see Theorem 7 in p. 109 of [3].
(iii) If there exist linear centralizers $\mathcal{Z}_{B_i} \in \mathfrak{C}(\mathcal{X})$ for $i = 1, \ldots, n$ where the matrices B_i are linearly independent and semi-simple (diagonalizable over $\mathbb{C}$), then $\mathcal{X}$ is analytically linearizable; see Theorem 7 in p. 109 of [3].

In this work we focus on the interplay between the structure of the normalizers $\mathfrak{N}(\mathcal{X})$ and the property that $\mathcal{X}$ be analytically orbitally linearizable. The following theorem generalizes one result of [5] in the planar case $n = 2$ to arbitrary dimension and to a more general linear part of the normalizer involved.

Theorem 1. *Consider the analytic vector field $\mathcal{X} = \mathcal{X}_A + \cdots$ in $\mathbb{R}^n$ with $A \neq 0$. Then $\mathcal{X}$ is analytically orbitally linearizable if, and only if, there exists an analytic normalizer $\mathcal{Z} \in \mathfrak{N}(\mathcal{X})$ possessing an analytic eigenfactor and having the form $\mathcal{Z} = \mathcal{Z}_B + \cdots$ where the spectrum $\sigma(B) = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{C}$ is nonresonant and satisfies either* $\mathrm{Re}(\lambda_i) > 0$ *or* $\mathrm{Re}(\lambda_i) < 0$ *for all i.*

A necessary condition for analytical orbital linearization using n independent normalizers is given in the next result.

Theorem 2. *If $\mathcal{X}$ is analytically orbitally linearizable, then there are normalizers $\mathcal{Z}_i \in \mathfrak{N}(\mathcal{X})$ for $i = 1, \ldots, n$ with analytic eigenfactors and such that $\mathcal{Z}_i = \mathcal{Z}_{B_i} + \cdots$ where the matrices B_i are linearly independent.*

Remark 3. From the proof of Theorem 2 we know that the relations $[\mathcal{X}, \mathcal{Z}_i] = \Lambda_i(x)\mathcal{X}$ satisfy $\Lambda_i(0) = 0$. Moreover, $\Phi_*\mathcal{Z}_i$ are linear vector fields where Φ is an analytic normalization that brings $\mathcal{X}$ to its orbitally linearizable normal form.

The following result gives the structure of the set $\mathfrak{N}(\mathcal{X})$ provided that $\mathcal{X}$ is analytically orbitally linearizable around the origin. Here a meromorphic function will be a function that can be expressed as the quotient of two real analytic functions around the origin.

Corollary 4. *Let $\mathcal{X} = \mathcal{X}_A + \cdots$ with $A \neq 0$ be analytically orbitally linearizable. Then $\mathcal{Z} \in \mathfrak{N}(\mathcal{X})$ if and only if there are local meromorphic functions μ and μ_i such that $\mathcal{Z} = \sum_{i=1}^{n-1} \mu_i\mathcal{Z}_i + \mu\mathcal{X}$ is analytic with $\mathcal{Z}_i \in \mathfrak{N}(\mathcal{X})$ and $\mathcal{X}(\mu_i) \equiv 0$ for $i = 1, \ldots, n-1$.*

The paper is organized as follows. In Section 2 we present detailed proofs of all the former results. The work ends with Section 3 where we emphasize that some of the results of this work can be generalized in a natural way from the analytic setting to the formal power series scenario. In fact all the results of the work can be restated for the more general category of holomorphic vector fields in $\mathbb{C}^n$ instead of the restriction to real analytic vector fields.