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Constraints on automorphism groups of higher dimensional manifolds

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ABSTRACT

In this note, we prove, for instance, that the automorphism group of a rational manifold X which is obtained from $\mathbb{P}^k(\mathbf{C})$ by a finite sequence of blow-ups along smooth centers of dimension at most r with $k > 2r + 2$ has finite image in $GL(H^*(X, \mathbf{Z}))$. In particular, every holomorphic automorphism $f : X \to X$ has zero topological entropy.

1. Introduction

1.1. Dimensions of indeterminacy loci

Recall that a rational map admitting a rational inverse is called birational. Birational transformations are, in general, not defined everywhere. The domain of definition of a birational map $f : M \to N$ is the largest Zariski-open subset on which f is locally a well defined morphism. Its complement is the indeterminacy set $\mathrm{Ind}(f)$; its codimension is always larger than, or equal to, 2. The following statement shows that the dimension of $\mathrm{Ind}(f)$ and $\mathrm{Ind}(f^{-1})$ cannot be too small simultaneously unless f is an automorphism. This result is inspired by a nice argument of Nessim Sibony concerning the degrees of regular automorphisms of the complex affine space $\mathbf{C}^k$ (see [13]). It may be considered as an extension of a theorem due to Matsusaka and Mumford (see [10], and [7, Exercise 5.6]).

Theorem 1.1. *Let $\mathbf{k}$ be a field. Let M be a smooth connected projective variety defined over $\mathbf{k}$. Let f be a birational transformation of M. Assume that the following two properties are satisfied.*

(i) *the Picard number of M is equal to 1;*
(ii) *the indeterminacy sets of f and its inverse satisfy*

$$\dim(\mathrm{Ind}(f)) + \dim(\mathrm{Ind}(f^{-1})) < \dim(M) - 2.$$

Then f is an automorphism of M.

Moreover, $\mathrm{Aut}(M)$ is an algebraic group because the Picard number of M is equal to 1. As explained below, this statement provides a direct proof of the following corollary, which was our initial motivation.

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Corollary 1.2. *Let M_0 be a smooth, connected, projective variety with Picard number* 1*. Let m be a positive integer, and $\pi_i : M_{i+1} \to M_i$, $i = 0, \ldots, m-1$, be a sequence of blow-ups of smooth irreducible subvarieties of dimension at most r. If $\dim(M_0) > 2r + 2$ then the number of connected components of $\mathsf{Aut}(M_m)$ is finite; moreover, the projection $\pi : M_m \to M_0$ conjugates $\mathsf{Aut}(M_m)$ to a subgroup of the algebraic group $\mathsf{Aut}(M_0)$.*

For instance, if M_0 is the projective space (respectively a cubic hypersurface of $\mathbb{P}^4_{\mathbf{k}}$) and if one modifies M_0 by a finite sequence of blow-ups of points, then $\mathsf{Aut}(M_0)$ is isomorphic to a linear algebraic subgroup of $\mathrm{PGL}_4(\mathbf{k})$ (respectively is finite). This provides a sharp (and strong) answer to a question of Eric Bedford. In Section 3, we provide a second, simpler proof of this last statement.

Remark 1.3. The initial question of E. Bedford concerned the existence of automorphisms of compact Kähler manifolds with positive topological entropy in dimension > 2. This link with dynamical systems is described, for instance, in [4]. If a compact complex surface S admits an automorphism with positive entropy, then S is Kähler and is obtained from the projective plane $\mathbb{P}^2(\mathbf{C})$, a torus, a $K3$ surface or an Enriques surface, by a finite sequence of blow-ups (see [5,6,12]). Examples of automorphisms with positive entropy are easily constructed on tori, K3 surfaces, or Enriques surfaces. Examples of automorphisms with positive entropy on rational surfaces are given in [2,3,11]; these examples are obtained from birational transformations f of the plane by a finite sequence of blow-ups that resolves all indeterminacies of f and its iterates simultaneously. These results suggest looking for birational transformations of $\mathbb{P}^n_{\mathbf{C}}$, $n \geq 3$, that can be lifted to automorphisms with a nice dynamical behavior after a finite sequence of blow-ups; the above result shows that at least one center of the blow-ups must have dimension $\geq n/2 - 1$.

Remark 1.4. Recently, Tuyen Truong obtained results which are similar to Corollary 1.2, but with hypothesis on the Hodge structure and nef classes of M_0 that replace our strong hypothesis on the Picard number (see [14,15]).

2. Dimensions of Indeterminacy loci

In this section, we prove Theorem 1.1 under a slightly more general assumption. Indeed, we replace assumption (i) with the following assumption

(i') There exists an ample line bundle L such that $f^*(L) \cong L^{\otimes d}$ for some $d > 1$.

This property is implied by (i). Indeed, if M has Picard number 1, the torsion-free part of the Néron–Severi group of M is isomorphic to $\mathbf{Z}$, and is generated by the class $[H]$ of an ample divisor H. Thus, $[f^*H]$ must be a multiple of $[H]$.

In what follows, we assume that f satisfies property (i') and property (ii). Replacing H by a large enough multiple, we may and do assume that H is very ample. Thus, the complete linear system $|H|$ provides an embedding of M into some projective space $\mathbb{P}^n_{\mathbf{k}}$, and we identify M with its image in $\mathbb{P}^n_{\mathbf{k}}$. With such a convention, members of $|H|$ correspond to hyperplane sections of M.

2.1. Degrees

Denote by k the dimension of M, and by $\deg(M)$ its degree, i.e. the number of intersections of M with a generic subspace of dimension $n - k$.

If $H_1, \ldots, H_k$ are hyperplane sections of M, and if $f^*(H_1)$ denotes the total transform of H_1 under the action of f, one defines the degree of f by the following intersection of divisors of M

$$\deg(f) = \frac{1}{\deg(M)} f^*(H_1) \cdot H_2 \cdots H_k.$$

Since M has Picard number 1, we know that divisor class $[f^*(H_1)]$ is proportional to $[H]$. Our definition of $\deg(f)$ implies that $f^*[H_1] = \deg(f)[H_1]$. As a consequence,

$$f^*(H_1) \cdot f^*(H_2) \cdots f^*(H_j) \cdot H_{j+1} \cdots H_k = \deg(f)^j \deg(M)$$

for all $0 \leq j \leq k$.

2.2. Degree bounds

Assume that the sum of the dimension of $\mathrm{Ind}(f)$ and of $\mathrm{Ind}(f^{-1})$ is at most $k - 3$. Then there exist at least two integers $l \geq 1$ such that

$$\dim(\mathrm{Ind}(f)) \leq k - l - 1;$$
$$\dim(\mathrm{Ind}(f^{-1})) \leq l - 1.$$

Let $H_1, \ldots, H_l$ and $H'_1, \ldots, H'_{k-l}$ be generic hyperplane sections of M; by Bertini's theorem,