J. Math. Anal. Appl. 405 (2013) 227–239

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Positive solutions of the Dirichlet problem for the prescribed mean curvature equation in Minkowski space ☆

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ARTICLE INFO

Article history:
Received 17 August 2012
Available online 8 April 2013
Submitted by J. Mawhin

Keywords:
Mean curvature
Minkowski space
Quasilinear elliptic equation
Dirichlet boundary condition
Positive solution
Existence
Multiplicity
Non-existence
Topological degree
Critical point theory

ABSTRACT

We prove the existence of multiple positive solutions of the Dirichlet problem for the prescribed mean curvature equation in Minkowski space

$$\begin{cases} -\mathrm{div}\left(\nabla u/\sqrt{1-|\nabla u|^2}\right) = f(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here Ω is a bounded regular domain in $\mathbb{R}^N$ and the function $f = f(x,s,\xi)$ is either sublinear, or superlinear, or sub-superlinear near $s = 0$. The proof combines topological and variational methods.

1. Introduction

Hypersurfaces of prescribed mean curvature in Minkowski space, with coordinates $(x_1, \dots, x_N, t)$ and metric $\sum_{i=1}^N dx_i^2 - dt^2$, are of interest in differential geometry and in general relativity. In this paper we are concerned with the existence of such a kind of hypersurfaces which are graphs of solutions of the Dirichlet problem:

$$\begin{cases} -\mathrm{div}\left(\nabla u/\sqrt{1-|\nabla u|^2}\right) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

We assume throughout that Ω is a bounded domain in $\mathbb{R}^N$, with a boundary $\partial\Omega$ of class C^2, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions. By a solution of (1) we mean a function $u \in W^{2,r}(\Omega)$, for some $r > N$, with $\|\nabla u\|_\infty < 1$, which satisfies the equation a.e. in Ω and vanishes on $\partial\Omega$. These are strong strictly spacelike solutions of (1) according to the terminology of, e.g., [5,15,2,10].

In [2,10] some general solvability results for (1) were proved under the assumption that the function f is globally bounded. Yet, as all spacelike solutions are uniformly bounded by the quantity $\frac{1}{2}\mathrm{d}(\Omega)$, with $\mathrm{d}(\Omega)$ the diameter of Ω, one can always reduce to that situation by truncation. Nevertheless it should be observed that if one already knows that problem (1) admits

☆ Research supported by M.I.U.R., in the frame of the P.R.I.N. "Equazioni differenziali ordinarie e applicazioni", and by University of Trieste, in the frame of the F.R.A. project "Equazioni differenziali ordinarie: aspetti qualitativi e numerici".

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0022-247X/$ – see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.jmaa.2013.04.003

zero as a solution, the results in [2,10] provide no further information. Therefore it may be interesting to investigate in such cases the existence of non-trivial, in particular positive, solutions. We point out that while this topic has been largely discussed in the literature for the Dirichlet problem associated with various classes of semilinear and quasilinear elliptic equations (including the prescribed mean curvature equation in Euclidean space), no result seems to be available for problem (1), at least when Ω is a general domain in $\mathbb{R}^N$.

Our aim here is indeed to extend to a genuine PDE setting what has been obtained in [6], for the one-dimensional problem, and in [3,4,7], for the radially symmetric problem in a ball. Namely, we will discuss the existence and the multiplicity of positive solutions of (1), assuming that the function $f = f(x, s)$ is sublinear, or superlinear, or sub-superlinear near $s = 0$.

In order to describe our results in a simple fashion, let us write the function f in the form

$$f(x, s) = \lambda\, a(x)(s^+)^p + \mu\, b(x)(s^+)^q, \tag{2}$$

where λ, μ are non-negative real parameters, $a, b : \bar{\Omega} \to \mathbb{R}$ are continuous functions, and p, q are given exponents satisfying $0 < p \leq 1 < q$. The coefficients a, b are assumed to be simultaneously positive at some point of Ω, but they are allowed to vanish in parts of Ω or to change sign. The following conclusions are then obtained.

Take $\mu = 0$ in (2). If the exponent $p \in]0, 1[$ is fixed, we prove that (1) has a positive solution for every $\lambda > 0$. If $p = 1$, we show that (1) has a positive solution for all large $\lambda > 0$, whereas non-existence of positive solutions is shown to occur for all sufficiently small $\lambda > 0$. It is immediately seen that in both cases the existence of positive solutions is guaranteed, with the same choices of λ, for any given $\mu > 0$.

Next, take $\lambda = 0$ in (2). If the exponent $q \in]1, +\infty[$ is fixed, we prove that (1) has at least two positive solutions for all large $\mu > 0$. Non-existence of positive solutions is also established for all sufficiently small $\mu > 0$.

Lastly, take $\lambda > 0$ and $\mu > 0$ in (2). Let the exponents $p \in]0, 1[$ and $q \in]1, +\infty[$ be given. Then (1) has at least three positive solutions for every large $\mu > 0$ and all sufficiently small $\lambda > 0$.

We point out that in all these statements no restriction is placed on the range of the exponent q.

Our results should be compared with similar ones obtained in [8] for a class of semilinear problems, and in [9] and in [14] for a class of quasilinear problems driven by the p-Laplace operator and the mean curvature operator in Euclidean space, respectively. In these papers some kinds of local analogues to the classical conditions of "sublinearity" and of "superlinearity" have been introduced, extending in various directions some of the results proved in the celebrated work by Ambrosetti, Brezis and Cerami [1]. We observe however that the multiplicity and the non-existence results we obtain for (1) are peculiar of this problem, due to the specific structure of the differential operator, and have no analogue in all the above mentioned cases.

We remark that, unlike in [6,7], our approach here is topological. This allows us to introduce a dependence on the gradient of the solution into the right-hand side f of the equation so that we can replace (1) with

$$\begin{cases} -\mathrm{div}\left(\nabla u/\sqrt{1 - |\nabla u|^2}\right) = f(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where again $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies the Carathéodory conditions. Of course, this problem does not have a variational structure anymore. However, our construction of the open sets, where we evaluate the degree of the solution operator associated with (3), relies on the knowledge of the radially symmetric solutions of suitable comparison problems, whose existence is proved by a minimization argument in [7].

We finally notice that the solvability of problem (3) has been explicitly raised as an open question in the recent work [13].

Notation. We list some additional notation that will be used throughout this paper. For $s \in \mathbb{R}$ we write $s^+ = \max\{s, 0\}$ and $s^- = -\min\{s, 0\}$. We denote by $B_R(x_0)$, or simply by B if no disambiguation is needed, the open ball in $\mathbb{R}^N$ centered at x_0 and having radius R. For functions $u, v : E \to \mathbb{R}$, with E a subset of $\mathbb{R}^N$ having positive measure, we write $u \leq v$ (in E) if $u(x) \leq v(x)$ a.e. in E, and $u < v$ (in E) if $u \leq v$ and $u(x) < v(x)$ in a subset of E having positive measure. A function u such that $u > 0$ is called positive. Assume that $\mathcal{O}$ is an open bounded set with a boundary $\partial\mathcal{O}$ of class C^1; for functions $u, v \in C^1(\bar{\mathcal{O}})$, we write $u \ll v$ (in $\bar{\mathcal{O}}$) if $u(x) < v(x)$ for every $x \in \mathcal{O}$ and, if $u(x) = v(x)$ for some $x \in \partial\mathcal{O}$, then $\frac{\partial v}{\partial \nu}(x) < \frac{\partial u}{\partial \nu}(x)$, where $\nu = \nu(x)$ denotes the unit outer normal to $\mathcal{O}$ at $x \in \partial\mathcal{O}$. A function u such that $u \gg 0$ is called strictly positive. We also set $C^1_0(\bar{\mathcal{O}}) = \{u \in C^1(\bar{\mathcal{O}}) : u = 0 \text{ on } \partial\mathcal{O}\}$. Finally, we denote by $\mathcal{I}$ the identity operator.

2. Preliminaries

We collect in this section some results that will be repeatedly used in the proof of our main result. We start with a comparison principle, which is a direct consequence of [2, Lemma 1.2].

Lemma 2.1. *Assume that $\mathcal{O}$ is a bounded domain in $\mathbb{R}^N$, with a boundary $\partial\mathcal{O}$ of class C^1. Suppose that $v_1, v_2 \in L^\infty(\mathcal{O})$ satisfy $v_1 \leq v_2$. Let, for $i = 1, 2$, $u_i \in W^{2,r}(\mathcal{O})$, for some $r > N$, be such that $\|\nabla u_i\|_\infty < 1$ and*

$$-\mathrm{div}\left(\nabla u/\sqrt{1 - |\nabla u|^2}\right) = v_i \quad \text{a.e. in } \mathcal{O}.$$