J. Math. Anal. Appl. 404 (2013) 105–114

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Multivalued variational inequalities and coincidence point results☆

Szilárd László

Department of Mathematics, Technical University of Cluj-Napoca, Str. Memorandumului nr. 28, 400114 Cluj - Napoca, Romania

ARTICLE INFO

Article history:
Received 7 May 2012
Available online 13 March 2013
Submitted by A. Dontchev

Keywords:
Operator of type ql
Multivalued variational inequality
KKM mapping
Fan's minimax theorem
Coincidence point

ABSTRACT

In this paper, we establish some existence results of the solutions for several multivalued variational inequalities involving elements belonging to a class of operators that was recently introduced in the literature. As applications we obtain some new coincidence point results in Hilbert spaces.

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1. Introduction

A considerable number of results that guarantee the existence of coincidence points for pairs of mappings are based on some generalizations of the Banach contraction principle (see for instance [1,3] and the references therein). In this paper we obtain some coincidence point results in Hilbert spaces without making use of any generalized contraction mapping. Our results are based on the existence of solutions of some variational inequalities involving operators belonging to a class, the class of operators of type ql that was recently introduced in [14]. Moreover, we show by an example that our results fail outside of this class.

Let X and Y be two arbitrary sets and $f : X \longrightarrow Y$, $T : X \rightrightarrows Y$ be two given mappings. We say that a point $x \in X$ is a coincidence point of f and T if $f(x) \in T(x)$.

Coincidence theory (the study of coincidence points) is, in most settings, a generalization of fixed point theory, the study of points x with $x \in T(x)$. Indeed, a fixed point is the special case obtained from the coincidence point by letting $X = Y$ and taking f to be the identity mapping. We show in the last section, that Kakutani's fixed point theorem is a particular case of our coincidence point results.

The variational inequality theory provides very powerful techniques for studying problems arising in mechanics, optimization, transportation, economics equilibrium, contact problems in elasticity, and other branches of mathematics (see, for instance [9,19,22,24]).

In recent years, many generalizations of variational inequalities have been considered, studied and applied in various directions (see, for instance [2,15,20,23,24]).

Let H be a real Hilbert space and let $C(H)$ be the family of all nonempty compact subsets of H. Let $T : H \longrightarrow C(H)$ be a set-valued operator, and let $f : H \longrightarrow H$ be a single-valued operator. Let K be a nonempty, closed, and convex set in H.

☆ This research was supported by a grant of the Romanian National Authority for Scientific Research CNCS – UEFISCDI, project number PN-II-RU-PD-2012-3-0166.

E-mail addresses: laszlosziszi@yahoo.com, laszlo_szilard@email.ro.

0022-247X/$ – see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.jmaa.2013.03.007

Consider the problem of finding $x \in H,\ f(x) \in K,\ u \in T(x)$ such that

$$\langle u, f(y) - f(x)\rangle \geq 0, \quad \forall f(y) \in K.$$

This problem is called a multivalued variational inequality. It has been shown, that a wide class of multivalued odd-order and nonsymmetric free, obstacle, moving equilibrium and optimization problems arising in pure and applied sciences can be studied via the multivalued variational inequality (see [20]).

In what follows we extend this problem to Banach spaces. Let X be a real Banach space and X^* be the topological dual of X. We denote by $\langle x^*, x\rangle$ the value of the linear and continuous functional $x^* \in X^*$ in $x \in X$. Let $K \subseteq X$ be nonempty and let $T : K \rightrightarrows X^*$ and $f : K \longrightarrow X$ be given operators. Consider the following problems. Find an element $x \in K$, such that

$$\forall\, y \in K\ \exists\, u \in T(x) : \langle u, f(y) - f(x)\rangle \geq 0, \tag{1}$$

$$\exists\, u \in T(x) : \forall\, y \in K\ \langle u, f(y) - f(x)\rangle \geq 0, \tag{2}$$

$$\forall\, y \in K\ \forall\, v \in T(y) : \langle v, f(y) - f(x)\rangle \geq 0. \tag{3}$$

It can be easily observed that if T is single valued, then (1), respectively (2) reduce to the general variational inequality of Stampacchia type, $VI_S(T, f, K)$, which consists in finding an element $x \in K$, such that $\langle T(x), f(y) - f(x)\rangle \geq 0$, for all $y \in K$ (see [14]). Let us denote by $S_w(T, f, K)$, respectively by $S(T, f, K)$ the set of solutions of (1), respectively the set of solutions of (2).

It is also obvious that if T is single valued, then (3) reduces to the general variational inequality of Minty type, $VI_M(T, f, K)$, which consists in finding an element $x \in K$, such that $\langle T(y), f(y) - f(x)\rangle \geq 0$, for all $y \in K$ (see [14]). Let us denote by $M(T, f, K)$ the set of solutions of (3).

In the present paper we give some existence results for the solutions of the problems (1)–(3). Beside our applications to coincidence point results, existence results for problems (1)–(3) will be helpful in the sense that one would like to know if a solution of problems (1)–(3) exists before one actually devises some plausible algorithms for solving the problems (1)–(3). The paper is organized as follows. In Section 2 we state some preliminary results that will be used throughout this paper. We state and prove a useful adaptation of KKM principle, in Banach spaces. In Section 3 we prove the main results of this paper concerning existence results of solution for the problems (1)–(3). By an example we show, that our results are the best possible in some sense, that is, if we drop the assumption that the operators f is of type ql in the hypothesis of our main theorems their conclusion fail. Finally, as applications of the results obtained, in Section 4 we provide some coincidence point results in Hilbert spaces.

2. Preliminaries

In order to continue our analysis we need the following notion. Let X_1, respectively X_2 be Hausdorff topological spaces and let $T : X_1 \rightrightarrows X_2$ be a set-valued operator with nonempty values. T is said to be upper semicontinuous if, for every $x_0 \in X_1$ and for every open set N containing $T(x_0)$, there exists a neighborhood M of x_0 such that $T(M) \subseteq N$.

We have the following characterization of upper semicontinuity (see [18]).

Lemma 2.1. *If T is compact-valued, then T is upper semicontinuous if and only if, for every net $(x_i) \subseteq X_1$ such that $x_i \longrightarrow x_0 \in X_1$ and for every $z_i \in T(x_i)$, there exist $z_0 \in T(x_0)$ and a subnet (z_{i_j}) of (z_i) such that $z_{i_j} \longrightarrow z_0$ (see also [8]). If X_1, respectively X_2 are metric spaces, instead of nets one can consider sequences (see [21]).*

Let X and Y be two Banach spaces. Recall that an operator $T : X \longrightarrow Y$ is called weak to norm-sequentially continuous at $x \in X$, if for every sequence (x_n) that converges weakly to x, we have that $T(x_n)$ converges to $T(x)$ in the topology of the norm of Y. An operator $T : X \rightrightarrows Y$ is said to be weak to weak* upper semicontinuous if, for every $x_0 \in X$ and for every open set $N \subseteq Y$, in the weak* topology of Y, containing $T(x_0)$, there exists a neighborhood M of x_0, in the weak topology of X, such that $T(M) \subseteq N$.

The following results will be very useful in the proof of our main existence results in the next section.

Lemma 2.2. *If $P \subset Q \subset X$, where Q is weakly compact and P is weakly sequentially closed then P is weakly compact.*

Proof. Indeed, by Eberlein–Šmulian theorem (see, for instance, [5]), Q is weakly sequentially compact. Let $(x_n) \subseteq P$, hence $(x_n) \subseteq Q$, which is weakly sequentially compact. Hence, there exists $(x_{n_k}) \subseteq (x_n)$, weakly convergent to a point $x \in Q$. But obviously $(x_{n_k}) \subseteq P$, which is weakly sequentially closed, hence $x \in P$. Thus P is weakly sequentially compact and according to Eberlein–Šmulian theorem P is weakly compact. □

Lemma 2.3. *Consider a bounded net $\left((x_i, x_i^*)\right)_{i\in I} \subset X \times X^*$, and assume that one of the following conditions is fulfilled:*

(a) *$x_i \rightharpoonup x$, i.e. the net (x_i) converges to x in the weak topology of X, and $x_i^* \longrightarrow x^*$, i.e. the net (x_i^*) converges to x^* in the topology of norm of X^*.*
(b) *$x_i \longrightarrow x$, i.e. the net (x_i) converges to x in the topology of norm of X, and $x_i^* \rightharpoonup^* x^*$, i.e. the net (x_i^*) converges to x^* in the weak * topology of X^*.*

Then $\left\langle x_i^*, x_i\right\rangle \longrightarrow \langle x^*, x\rangle$.

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