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Application of a composition of generating functions for obtaining explicit formulas of polynomials

Dmitry V. Kruchinin *, Vladimir V. Kruchinin

Tomsk State University of Control Systems and Radioelectronics (TUSUR), Tomsk, 634050, Russia

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ABSTRACT

Using notions of composita and composition of generating functions, we obtain explicit formulas for the Chebyshev polynomials, the Legendre polynomials, the Gegenbauer polynomials, the Associated Laguerre polynomials, the Stirling polynomials, the Abel polynomials, the Bernoulli Polynomials of the Second Kind, the Generalized Bernoulli polynomials, the Euler Polynomials, the Peters polynomials, and the Narumi polynomials.

1. Introduction

A polynomial is a mathematical expression involving a sum of powers of one or more variables multiplied by coefficients. The use of polynomials in other areas of mathematics is very impressive: continued fractions, operator theory, analytic functions, interpolation, approximation theory, numerical analysis, electrostatics, statistical quantum mechanics, special functions, number theory, combinatorics, stochastic processes, sorting and data compression, etc.. In this paper we use the method of obtaining expressions for polynomials based on the composition of generating functions, which was presented by the authors in [9].

The generating functions have an important role in many branches of mathematics and mathematical physics. In the literature one can find extensive investigations related to the generating functions for many polynomials [12,13,11,5,2].

In this paper we obtain explicit formulas for the Chebyshev polynomials, the Legendre polynomials, the Gegenbauer polynomials, the Associated Laguerre polynomials, the Stirling polynomials, the Abel polynomials, the Bernoulli Polynomials of the Second Kind, the Generalized Bernoulli polynomials, the Euler Polynomials, the Peters polynomials, and the Narumi polynomials.

2. Preliminary

In the paper [8], the authors introduced the notion of the *composita* of a given ordinary generating function $F(t) = \sum_{n>0} f(n)t^n$.

Suppose $F(t) = \sum_{n>0} f(n)t^n$ is the generating function, in which there is no free term $f(0) = 0$. From this generating function we can write the following condition

$$[F(t)]^k = \sum_{n>0} F(n,k)t^n. \tag{1}$$

The expression $F(n,k)$ is the *composita*, and it is denoted by $F^{\Delta}(n,k)$.

* Corresponding author.
E-mail addresses: kruchinindm@gmail.com, kdv@keva.tusur.ru (D.V. Kruchinin), kru@ie.tusur.ru (V.V. Kruchinin).

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Table 1
Examples of generating functions and their compositae.

Generating function $G(x)$	Composita $G^{\Delta}(n,k)$
$at+bt^2$	$a^{2k-n}b^{n-k}\binom{k}{n-k}$
$\frac{bt}{1-ax}$	$\binom{n-1}{k-1}a^{n-k}b^k$
$\ln(1+t)$	$\frac{k!}{n!}s(n,k)$
e^t-1	$\frac{k!}{n!}S(n,k)$

The *composita* is the function of two variables defined by

$$F^{\Delta}(n,k)=\sum_{\pi_k\in C_n} f(\lambda_1)f(\lambda_2)\cdots f(\lambda_k), \tag{2}$$

where C_n is a set of all compositions of an integer n, π_k is the composition $\sum_{i=1}^{k}\lambda_i=n$ into k parts exactly.

Comtet [4, p. 141] considered similar objects and identities for exponential generating functions, and called them potential polynomials. In this paper we consider the more general case of generating functions ordinary generating functions.

The expression $F^{\Delta}(n,k)$ takes a triangular form

$$\begin{array}{ccccccccccc}
 & & & & & F^{\Delta}_{1,1} & & & & & \\
 & & & & F^{\Delta}_{2,1} & & F^{\Delta}_{2,2} & & & & \\
 & & & F^{\Delta}_{3,1} & & F^{\Delta}_{3,2} & & F^{\Delta}_{3,3} & & & \\
 & & F^{\Delta}_{4,1} & & F^{\Delta}_{4,2} & & F^{\Delta}_{4,3} & & F^{\Delta}_{4,4} & & \\
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
F^{\Delta}_{n,1} & & F^{\Delta}_{n,2} & & \cdots & & \cdots & & F^{\Delta}_{n,n-1} & & F^{\Delta}_{n,n}.
\end{array}$$

For instance, we obtain the composita of the generating function

$$G(t,a,b)=at+bt^2.$$

Raising this generating function to the power of k and applying the binomial theorem, we obtain

$$[G(t,a,b)]^k=t^k(a+bt)^k=t^k\sum_{m=0}^{k}\binom{k}{m}a^{k-m}b^mt^m.$$

Substituting n for $m+k$, we get the following expression

$$[G(t,a,b)]^k=\sum_{n=k}^{2k}\binom{k}{n-k}a^{2k-n}b^{n-k}t^n=\sum_{n=k}^{2k}G^{\Delta}(n,k,a,b)t^n.$$

Therefore, the composita is

$$G^{\Delta}(n,k,a,b)=\binom{k}{n-k}a^{2k-n}b^{n-k}. \tag{3}$$

In the Table 1 we show a few compositae of known generating functions [4,17].

Here $s(n,k)$ and $S(n,k)$ stand for the Stirling numbers of the first kind and of the second kind, respectively (see [4,7]).

The Stirling numbers of the first kind $s(n,k)$ count the number of permutations of n elements with k disjoint cycles. The Stirling numbers of the first kind are defined by the following generating function

$$\psi_k(x)=\sum_{n\geq k}s(n,k)\frac{x^n}{n!}=\frac{1}{k!}\ln^k(1+x).$$

The Stirling numbers of the second kind $S(n,k)$ count the number of ways to partition a set of n elements into k nonempty subsets. A general formula for the Stirling numbers of the second kind is given as follows

$$S(n,k)=\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n.$$

The Stirling numbers of the second kind are defined by the following generating function

$$\Phi_k(x)=\sum_{n\geq k}S(n,k)\frac{x^n}{n!}=\frac{1}{k!}(e^x-1)^k.$$

Next we show some operations with compositae (for details see [8,10]).