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Subhomogeneity and subadditivity of the L^p-norm like functionals

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ABSTRACT

For a measure space (Ω, Σ, μ) denote by $S = S(\Omega, \Sigma, \mu)$ the set of all μ-integrable simple functions $x : \Omega \to \mathbb{R}$. For a bijection $\varphi : (0, \infty) \to (0, \infty)$ we consider the functional $\mathbf{P}_\varphi : S \to [0, \infty)$,

$$\mathbf{P}_\varphi(x) := \varphi^{-1}\left(\int_{\Omega(x)} \varphi \circ |x| \, d\mu\right),$$

where $\Omega(x)$ is the support of $x \in S$. One of the results says that if the measure μ has values in $(0, 1)$ and in $(1, \infty)$, the function φ is monotonic and $\mathbf{P}_\varphi$ satisfies the inequality

$$\mathbf{P}_\varphi(tx) \le t\mathbf{P}_\varphi(x), \quad t > 1, x \in S,$$

then φ is a power function. Some characterizations of the functions φ in two remaining cases when either $\mu(\Sigma) \cap (1, \infty) = \emptyset$ or $\mu(\Sigma) \cap (0, 1) = \emptyset$ are given. The subadditivity of $\mathbf{P}_\varphi$, i.e. a generalization of the Minkowski inequality, is also considered.

0. Introduction

For a measure space (Ω, Σ, μ) denote by $S = S(\Omega, \Sigma, \mu)$ the real linear space of all μ-integrable simple functions $x : \Omega \to \mathbb{R}$. For an arbitrary bijection $\varphi : (0, \infty) \to (0, \infty)$ we define the functional $\mathbf{P}_\varphi : S \to [0, \infty)$ by the formula

$$\mathbf{P}_\varphi(x) := \varphi^{-1}\left(\int_{\Omega(x)} \varphi \circ |x| \, d\mu\right),$$

where $\Omega(x)$ is the support of $x \in S$ such that $\mu(\Omega(x)) > 0$; if $\mu(\Omega(x)) = 0$ we put $\mathbf{P}_\varphi(x) := 0$. If $\varphi(t) = \varphi(1)t^p$ for some $p \ge 1$ then $\mathbf{P}_\varphi$ is the L^p-norm.

Assume that (Ω, Σ, μ) is nontrivial. Then, under a weak regularity condition on φ, the functional $\mathbf{P}_\varphi$ is (positively) *homogeneous*, that is

$$\mathbf{P}_\varphi(tx) = t\mathbf{P}_\varphi(x), \quad t > 0, \ x \in S,$$

if, and only if, φ is a power function, i.e. $\varphi(r) = \varphi(1) r^p$ for some real $p \neq 0$ [4].

In the present paper we examine conditions under which $\mathbf{P}_\varphi$ satisfies the inequality

$$\mathbf{P}_\varphi(tx) \le t\mathbf{P}_\varphi(x), \quad t > 1, \ x \in S,$$

that is, when $\mathbf{P}_\varphi$ is *subhomogeneous* (cf. [18,7]). The following result is proved in Section 2 (Theorem 2). If there are $A, B \in \Sigma$ such that

$$0 < \mu(A) < 1 < \mu(B) < \infty, \tag{1}$$

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and φ is monotonic, then $\mathbf{P}_\varphi$ is *subhomogeneous* if, and only if, φ is a power function, that is, for some $p \in \mathbb{R}$, $p \neq 0$,

$$\mathbf{P}_\varphi(x) = \left(\int_{\Omega(x)} |x|^p \, d\mu \right)^{1/p}, \quad x \in S;$$

in particular, $\mathbf{P}_\varphi$ is homogeneous.

Thus, under the monotonicity condition on φ, the subhomogeneous but not homogeneous functionals $\mathbf{P}_\varphi$ may exist only if the underlying measure space (Ω, Σ, μ) satisfies one of the following two conditions

(I): for any set $A \in \Sigma$, either $\mu(A) \leq 1$ or $\mu(A) = \infty$;
(II): for any set $A \in \Sigma$, either $\mu(A) = 0$ or $\mu(A) \geq 1$.

Any probability measure space is of type (I), and the counting measure space is of type (II).

For the measure spaces of type (I), in Section 3, we prove that if φ is increasing, and its right (left) derivative is geometrically concave, or φ is decreasing and its right (left) derivative is geometrically convex, then $\mathbf{P}_\varphi$ is subhomogeneous (Theorem 3), and, if the range of the measure (or the union of applied measures) is enough reach, then the converse implication holds true (Theorem 4).

For the measure spaces of type (II), in Section 4, we show that if φ is increasing and geometrically convex, or φ is decreasing and geometrically concave, then $\mathbf{P}_\varphi$ is subhomogeneous (Theorem 5) and, if the range of the measure (or the union of applied measures) is enough reach, then the converse implication holds true (Theorem 6).

In Section 5 we give sufficient and necessary and sufficient conditions for the subhomogeneity and subadditivity of the functional $\mathbf{P}_\varphi$.

The suitable properties of superhomogeneity and superadditivity of $\mathbf{P}_\varphi$ can be obtained by easy modifications of the obtained results (Final Remark).

Some, important for this paper, auxiliary results are presented in Section 1. In particular we show that if in a (nontrivial) measure space (Ω, Σ, μ) there is a set $A \in \Sigma$ such that $\mu(A) = 1$ and $\mathbf{P}_\varphi$ is subadditive in S, i.e.

$$\mathbf{P}_\varphi(x+y) \leq \mathbf{P}_\varphi(x) + \mathbf{P}_\varphi(y), \quad x, y \in S,$$

then φ must be increasing and convex (Theorem 1).

Let us mention that in [5] (cf. also [10–12]), under assumption (1) on the measure space, the converse of the Minkowski inequality theorem is proved.

1. Auxiliary results

Let $I \subset (0, \infty)$ be an interval. A function $f : I \to (0, \infty)$ is said to be *Jensen geometrically convex* (*Jensen geometrically concave*) if $f\left(\sqrt{xy}\right) \leq \sqrt{f(x)f(y)}$ for all $x, y \in I$, (the reversed inequality holds), and *Jensen geometrically affine* if the equality is satisfied.

A function $f : I \to (0, \infty)$ is said to be *geometrically convex* (*geometrically concave*) if

$$f\left(x^t y^{1-t}\right) \leq [f(x)]^t \, [f(y)]^{1-t}, \quad t \in (0,1), \; x, y \in I,$$

(the reversed inequality holds), and *geometrically affine*, if the equality is satisfied [7].

Moreover f is geometrically affine iff, $f(x) = ax^p$ $(x \in I)$ for some $p \in \mathbb{R}$ and $a > 0$.

Of course, a function $f : I \to (0, \infty)$ is (Jensen) geometrically convex iff $\log \circ f \circ \exp$ is Jensen convex on the interval $\log(I)$.

As an immediate consequence of the theory of convex functions (Kuczma [3]) we hence obtain the following.

Remark 1. Let f be Jensen geometrically convex. If f is continuous at least at one point, or bounded from above in a neighborhood of a point, or measurable, then f is geometrically convex.

For a function $f : I \to \mathbb{R}$ denote by f'_- the left-sided derivative (by f'_+ the right-sided derivative of f) if it exists.

Remark 2. Suppose that f is right-sided differentiable in I. Then f is geometrically convex (geometrically concave) if, and only if, the function

$$I \ni x \longmapsto \frac{f'_+(x)}{f(x)} x$$

is increasing (decreasing).

Indeed, f is geometrically convex iff $\log \circ f \circ \exp$ is convex, that is, iff the function

$$\frac{f'_+ \circ \exp}{f \circ \exp} \exp \quad \text{is increasing.}$$

Since exp is increasing, this function is increasing iff the function $I \ni x \longmapsto \frac{f'(x)}{f(x)} x$ is increasing.

Of course this remark remains true if we replace f'_+ by f'_-.