J. Math. Anal. Appl. 406 (2013) 400–418

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Log-convexity and log-concavity for series in gamma ratios and applications

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A R T I C L E I N F O

Article history:
Received 7 January 2013
Available online 2 May 2013
Submitted by Kathy Driver

Keywords:
Gamma function
Log-concavity
Log-convexity
q-log-concavity
Wright log-concavity
Turán inequality
Kummer function
Generalized hypergeometric function

A B S T R A C T

Polynomial sequence $\{P_m\}_{m\geq 0}$ is q-logarithmically concave if $P_m^2 - P_{m+1}P_{m-1}$ is a polynomial with nonnegative coefficients for any $m \geq 1$. We introduce an analogue of this notion for formal power series whose coefficients are nonnegative continuous functions of a parameter. Four types of such power series are considered where the parameter dependence is expressed by a ratio of gamma functions. We prove six theorems stating various forms of q-logarithmic concavity and convexity of these series. The main motivating examples for these investigations are hypergeometric functions. In the last section of the paper we present new inequalities for the Kummer function, the ratio of the Gauss functions and the generalized hypergeometric function obtained as direct applications of the general theorems.

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1. Introduction

We adopt the standard notation $\mathbb{N}$ for the set of positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}$ will stand for reals and $\mathbb{R}_+$ for nonnegative reals. The gamma function $\Gamma(x)$ was introduced by Leonard Euler who also demonstrated that its second logarithmic derivative is positive for positive values of x. In modern language this means that $\Gamma(x)$ is logarithmically convex (i.e. its logarithm is a convex function). A sum of log-convex functions can be shown to be log-convex using the Hölder inequality or a theorem of Montel [21, Theorem 1.4.5.2]. Additivity implies then that the (finite or infinite) sum $f(\mu;x) := \sum f_k\Gamma(\mu+k)x^k$ is the logarithmically convex function of μ for fixed $x \geq 0$ once the coefficients f_k are assumed to be nonnegative. It is not difficult to see that much more is true [17, Theorem 2]: the formal power series $f(\mu;x)f(\mu+\alpha+\beta;x) - f(\mu+\alpha;x)f(\mu+\beta;x)$ has nonnegative coefficients at all powers of x if $\alpha, \beta \geq 0$. In [13] we considered a similar problem for the series $g(\mu;x) := \sum g_k\{\Gamma(\mu+k)\}^{-1}x^k$. Here each term is a log-concave function of μ, so that lack of additivity of logarithmic concavity does not allow to draw any immediate conclusions about the sum. We have demonstrated, however, that the sequence $\{g(\mu;x)\}_{\mu\in\mathbb{N}}$ is log-concave for fixed $x > 0$ if the sequence of coefficients $\{g_k\}_{k\in\mathbb{N}}$ is log-concave and without internal zeros. Moreover, in this case $g(\mu;x)g(\mu+\alpha+\beta;x) - g(\mu+\alpha;x)g(\mu+\beta;x)$ has nonnegative coefficients at all powers of x if $\alpha, \beta \in \mathbb{N}$. The two sums above can be generalized naturally to series in product ratios of gamma functions having the form (3) below. Several known questions in financial mathematics [5,6], multidimensional statistics [29], probability [24] and special functions [3,4,12] reduce to or depend on log-convexity or log-concavity of special cases of such generalized series. Similar coefficient-wise positivity of product differences is also important in combinatorics. The following definition is attributed to Richard

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0022-247X/$ – see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.jmaa.2013.04.061

Stanley [27, p. 795]. A sequence of polynomials $\{P_m(q)\}_{m\geq 0}$ is said to be q-log-concave if

$$P_m(q)^2 - P_{m+1}(q)P_{m-1}(q)$$

is a polynomial with nonnegative coefficients for any $m \geq 1$. It is strongly q-log-concave if

$$P_m(q)P_n(q) - P_{m+1}(q)P_{n-1}(q)$$

is a polynomial with nonnegative coefficients for all $m \geq n \geq 1$. The latter notion was introduced by Sagan [27]. Many sequences of combinatorial polynomials especially those related to q-calculus possess these properties (see [8,27] for details and references). We will need extensions of these notions to families of formal power series. To be consistent with the standard definitions of log-concavity and Wright log-concavity [22, Chapter I.4], [25, Section 1.1] and to make our formulations more compact, we found it reasonable to change the combinatorial terminology slightly. We decided to keep the letter "q" in our definitions to retain a connection to combinatorial terminology, while the argument is changed to x to avoid confusion with q-calculus. Suppose

$$f(\mu; x) = \sum_{k=0}^{\infty} f_k(\mu)x^k \tag{1}$$

is a formal power series with nonnegative coefficients which depend continuously on a nonnegative parameter μ.

Definition. The family $\{f(\mu; x)\}_{\mu\geq 0}$ is **Wright q-log-concave** if formal power series

$$\phi_\mu(\alpha, \beta; x) := f(\mu + \alpha; x)f(\mu + \beta; x) - f(\mu; x)f(\mu + \alpha + \beta; x) \tag{2}$$

has nonnegative coefficients at all powers of x for all $\mu, \alpha, \beta \geq 0$. If this property only holds for $\alpha \in \mathbb{N}$ and all $\mu, \beta \geq 0$ we will say that $\{f(\mu; x)\}_{\mu\geq 0}$ is **discrete Wright q-log-concave**. Finally, $\{f(\mu; x)\}_{\mu\geq 0}$ is **discrete q-log-concave** if $\phi_\mu(\alpha, \beta; x)$ has nonnegative coefficients at all powers of x for $\alpha \in \mathbb{N}$, $\beta \geq \alpha - 1$ and all $\mu \geq 0$.

If each function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is associated with the family of formal power series $\{f(\mu; x)\}_{\mu\geq 0}$ with $f_0 = f(\mu)$ and zero coefficients at all positive powers of x, the above definitions become consistent with the following standard terminology: $\mu \to f(\mu)$ is called Wright log-concave on $\mathbb{R}_+$ if $f(\mu+\alpha)f(\mu+\beta) \geq f(\mu)f(\mu+\alpha+\beta)$ for all $\mu, \alpha, \beta \geq 0$ [22, Chapter I.4], [25, Definition 1.13]; it is discrete Wright log-concave on $\mathbb{R}_+$ if the above inequality holds for $\alpha \in \mathbb{N}$ and all $\mu, \beta \geq 0$ and discrete log-concave if it holds for $\alpha \in \mathbb{N}$, $\beta \geq \alpha - 1$ and $\mu \geq 0$ [13]. For continuous functions Wright log-concavity is equivalent to log-concavity (i.e. concavity of the logarithm). Discrete Wright log-concavity implies discrete log-concavity but not vice versa (see details in [13]). All above definitions also apply if we change "concave" to "convex", "non-negative" to "non-positive" and reverse the sign of all inequalities. In the theory of special functions discrete log-concavity and log-convexity are also frequently referred to as "Turán type inequalities" following the classical result of Paul Turán for Legendre polynomials [30]: $[P_n(x)]^2 > P_{n-1}(x)P_{n+1}(x)$, $-1 < x < 1$. Note, however, that the sequence $\{P_n(x)\}_{n\geq 0}$ is not q-log-concave. General Wright convex functions attracted a lot of attention recently (see, for instance, [11,18] and references therein) following a fundamental result of Ng [23].

If $f : \mathbb{N}_0 \to \mathbb{R}_+$ is a sequence, then discrete log-concavity reduces to inequality $f_k^2 \geq f_{k-1}f_{k+1}$, $k \in \mathbb{N}$. We will additionally require that the sequence $\{f_k\}_{k=0}^{\infty}$ is non-trivial and has no internal zeros, i.e. $f_N = 0$ implies either $f_{N+i} = 0$ for all $i \in \mathbb{N}_0$ or $f_{N-i} = 0$ for $i = 0, 1, \ldots, N$. Such sequences are also known as PF_2 (Pólya frequency sub two) or doubly positive [14]. Clearly, if f is (Wright) log-concave then $1/f$ is (Wright) log-convex. Notwithstanding the simplicity of this relation, several important properties of log-concavity and log-convexity differ. As we already mentioned above, log-convexity is preserved under addition while log-concavity is not. Further, log-convexity is a stronger property than convexity whereas log-concavity is weaker than concavity. Further properties of log-convex and log-concave functions can be found, for instance, in [19, 3E, 16D, 18B], [24, Chapter 2] and [25, Chapter 13].

The questions considered in [13,17] and in this paper are particular cases of the following general problem: under what conditions on a nonnegative sequence $\{f_k\}$ and the numbers a_i, b_j the series

$$f(\mu; x) = \sum_{k=0}^{\infty} f_k \frac{\prod_{i=1}^{n} \Gamma(a_i + \mu + \varepsilon_i k)}{\prod_{j=1}^{m} \Gamma(b_j + \mu + \varepsilon_{n+j} k)} x^k \tag{3}$$

is (discrete, Wright) q-log-concave or q-log-convex? Here ε_r can take value 1 or 0. In particular, if the ratio f_{k+1}/f_k is a rational function of k the series in (3) is hypergeometric (possibly times some gamma functions) and μ represents parameter shift [2, Chapter 2].

The following cases of (3) were treated in [17]: $n = 1, m = 0, \varepsilon_1 = 1$; $n = m = 1, \varepsilon_1 = 1, \varepsilon_2 = 0, a_1 = b_1$; and $n = m = 1, \varepsilon_1 = 0, \varepsilon_2 = 1, a_1 = b_1$. In [13] we handled $n = 0, m = 1, \varepsilon_1 = 1$. This paper is concerned with the following cases of (3):

(a) $n = m = 2, \varepsilon_1 = 1, \varepsilon_2 = 0, \varepsilon_3 = 0, \varepsilon_4 = 1, a_1 = b_1, a_2 = b_2$;

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