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Self-intersections of the Riemann zeta function on the critical line

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ABSTRACT

We show that the Riemann zeta function ζ has only countably many self-intersections on the critical line, i.e., for all but countably many $z \in \mathbb{C}$ the equation $\zeta(\frac{1}{2}+it) = z$ has at most one solution $t \in \mathbb{R}$. More generally, we prove that if F is analytic in a complex neighborhood of $\mathbb{R}$ and locally injective on $\mathbb{R}$, then either the set $\{(a, b) \in \mathbb{R}^2 : a \neq b \text{ and } F(a) = F(b)\}$ is countable, or the image $F(\mathbb{R})$ is a loop in $\mathbb{C}$.

1. Introduction

In the half-plane $\{s \in \mathbb{C} : \sigma > 1\}$ the *Riemann zeta function* is defined by the equivalent expressions

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}.$$

In the extraordinary memoir of Riemann [7] it is shown that ζ extends to a meromorphic function on the entire complex plane with its only singularity being a simple pole at $s = 1$, and it satisfies the functional equation relating its values at s and $1 - s$. The Riemann hypothesis asserts that every non-real zero of ζ lies on the *critical line*

$$\mathcal{L} := \left\{ s \in \mathbb{C} : \sigma = \frac{1}{2} \right\}.$$

By a *self-intersection of* ζ *on the critical line* we mean an element of the set

$$\{(s_1, s_2) \in \mathcal{L}^2 : s_1 \neq s_2 \text{ and } \zeta(s_1) = \zeta(s_2)\}.$$

Our aim in the present paper is to prove that this set is countable.

Theorem 1. *The Riemann zeta function has only countably many self-intersections on the critical line.*

In other words, the equation $\zeta(\frac{1}{2} + it) = z$ has at most one solution $t \in \mathbb{R}$ for all but countably many $z \in \mathbb{C}$. This complements the fact that $\zeta(\frac{1}{2} + it) = z$ has at least two solutions $t \in \mathbb{R}$ for infinitely many $z \in \mathbb{C}$, which follows from a

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recent result of Banks and Kang [1, Theorem 1.2]. Moreover, it has been conjectured in [1] that $\zeta(\frac{1}{2}+it)=z$ has no more than two solutions $t\in\mathbb{R}$ for every nonzero $z\in\mathbb{C}$; our Theorem 1 makes it clear that there are at most countably many counterexamples to this conjecture.

There are two main ingredients in the proof of Theorem 1. The first is that the curve $f(t):=\zeta(\frac{1}{2}+it)$ is *locally injective on* $\mathbb{R}$, i.e., for every $t\in\mathbb{R}$ there is an open real interval containing t on which f is one-to-one; this is Proposition 3 of Section 2. A result of Levinson and Montgomery [2] guarantees the local injectivity of f around points t for which $f(t)\neq 0$, and in Proposition 3 the local injectivity of f is also established around points t with $f(t)=0$.

The second ingredient in the proof of Theorem 1 is the following general result about self-intersections of locally injective analytic curves.

Theorem 2. *Let F be a function which is analytic in a complex neighborhood of the real line $\mathbb{R}$, and suppose that F is locally injective on $\mathbb{R}$. If the set of self-intersections*

$$\{(a,b)\in\mathbb{R}^2 : a\neq b \text{ and } F(a)=F(b)\}$$

is uncountable, then $F(\mathbb{R})$ is a loop in $\mathbb{C}$.

By a *loop in* $\mathbb{C}$ we mean the image of a continuous map $L:[\gamma,\delta]\to\mathbb{C}$ such that $L(\gamma)=L(\delta)$. Since every loop is compact, Theorem 2 applied with $F:=f$ immediately implies Theorem 1 in view of the fact that ζ is unbounded on the critical line (see, for example, [8, Theorem 8.12]).

The proof of Theorem 2 in Section 3 relies on intersection properties of analytic curves that were first discovered by Markushevich [3] and were later extended by Mohon'ko [4,5]; see Proposition 4 of Section 3 and the remarks that follow.

2. Local injectivity

Proposition 3. *The curve*

$$f(t):=\zeta\left(\frac{1}{2}+it\right)\quad(t\in\mathbb{R}) \tag{1}$$

is locally injective on $\mathbb{R}$.

Proof. For every $a\in\mathbb{R}$, let Σ_a denote the collection of open intervals $\mathcal{I}$ in $\mathbb{R}$ that contain a. For any fixed a we must show that f is one-to-one on an interval $\mathcal{I}\in\Sigma_a$.

In the case that $f(a)\neq 0$ we use a result of Levinson and Montgomery (see [2, p. 53]) which states that $\zeta(s)=0$ whenever $\zeta'(s)=0$ and $\sigma=\frac{1}{2}$. As $f(a)\neq 0$ we have $f'(a)\neq 0$, hence f is locally invertible in a complex neighborhood of a; in particular, f is one-to-one on some interval $\mathcal{I}\in\Sigma_a$.

Let $\mathcal{Z}$ denote the set of all zeros of f. If $t\notin\mathcal{Z}$, we define

$$\vartheta(t):=\arg f(t)=\Im\log\zeta\left(\frac{1}{2}+it\right) \tag{2}$$

by continuous variation of the argument from 2 to $2+it$ to $\frac{1}{2}+it$, starting with the value 0, and we denote by $N(t)$ the number of zeros $\rho=\beta+i\gamma$ of $\zeta(s)$ in the rectangle $0<\beta<1$, $0<\gamma<t$. Then

$$\vartheta(t)=\pi\cdot(N(t)-1)+g(t)\quad(t\notin\mathcal{Z}), \tag{3}$$

where

$$g(t):=-\arg\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)+\frac{t}{2}\log\pi\quad(t\in\mathbb{R});$$

see, for example, Montgomery and Vaughan [6, Theorem 14.1]. Then

$$g'(t)=-\frac{1}{2}\Re\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{it}{2}\right)+\frac{1}{2}\log\pi\quad(t\in\mathbb{R}), \tag{4}$$

and from the well known relation $(\Gamma'/\Gamma)'(s)=\sum_{n=0}^{\infty}(n+s)^{-2}$ we see that

$$g''(t)=-16t\sum_{n=0}^{\infty}\frac{4n+1}{((4n+1)^2+4t^2)^2}\quad(t\in\mathbb{R}).$$

Thus, if $\Theta:=6.2898\ldots$ is the unique positive root of the function defined by the right-hand side of (4), it is easy to see that g is strictly decreasing at any $t\in\mathbb{R}$ with $|t|>\Theta$.