J. Math. Anal. Appl. 406 (2013) 482–501

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Nonexistence of solutions for prescribed mean curvature equations on a ball

Hongjing Pan [a], Ruixiang Xing [b,*]

[a] School of Mathematical Sciences, South China Normal University, Guangzhou, 510631, China
[b] School of Mathematics and Computational Science, Sun Yat-sen University, Guangzhou, 510275, China

ARTICLE INFO

Article history:
Received 29 January 2013
Available online 9 May 2013
Submitted by Barbara Kaltenbacher

Keywords:
Prescribed mean curvature equation
Radial solution
Bistable
Time map
Quasilinear problem
Superlinear problem
Sublinear problem
Nonexistence
Nonlinear eigenvalue problem

ABSTRACT

We prove two nonexistence results of radial solutions to the prescribed mean curvature type problem on a ball

$$\begin{cases} -\operatorname{div}\left(\dfrac{Du}{\sqrt{1+|Du|^2}}\right) = \lambda f(u), & x \in \mathcal{B}_R \subseteq \mathbb{R}^n, \\ u = 0, & x \in \partial\mathcal{B}_R, \end{cases}$$

where λ is a positive parameter, f is a continuous function with $f(0) = 0$. Under suitable assumptions on f, we show that the problem with "superlinear" f has no nontrivial positive solutions for small λ while the problem with "sublinear" f has no nontrivial positive solutions for large λ. The former covers many well-known nonexistence results by Finn, Serrin, Narukawa and Suzuki, Ishimura, Pan and Xing. To the best of the authors' knowledge, the latter is the first nonexistence result involving sublinear mean curvature type equations in higher dimensions. In particular, the sublinear cases contain some important logistic type nonlinearities. These nonexistence results differ greatly from those of semilinear problems.

1. Introduction

Consider the following prescribed mean curvature equation

$$\begin{cases} -\operatorname{div}\left(\dfrac{Du}{\sqrt{1+|Du|^2}}\right) = \lambda f(u), & x \in \mathcal{B}_R, \\ u = 0, & x \in \partial\mathcal{B}_R, \end{cases} \tag{1.1}$$

where $\lambda > 0$ and $\mathcal{B}_R \subset \mathbb{R}^n (n \geqslant 2)$ is a ball of radius R. In this paper, we are concerned with the nonexistence of radial solutions of (1.1) when λ changes.

The motivation of the paper stems from the interesting bifurcation patterns of two one-dimensional problems. One is the case $f(u) = u^p$ $(p > 0)$, the other is the case $f(u) = e^u - 1$. For $n = 1$, these two examples have been well investigated in [15] and [29], respectively. The bifurcation curves are depicted in Fig. 1 and Fig. 2, where $\alpha = \max u = u(0)$, the continuous thick line represents classical solutions, the continuous thin curve is the gradient blow-up curve. It is observed from the diagrams that nonexistence of positive solutions in the sublinear case holds for large λ, while in superlinear cases holds for not too large λ (in some circumstances provided that R is small enough). It is natural to ask whether the similar nonexistence results still hold for general higher dimensional problems.

* Corresponding author.
E-mail addresses: panhj@scnu.edu.cn (H. Pan), xingrx@mail.sysu.edu.cn, xingruixiang@gmail.com (R. Xing).

0022-247X/$ – see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.jmaa.2013.05.003

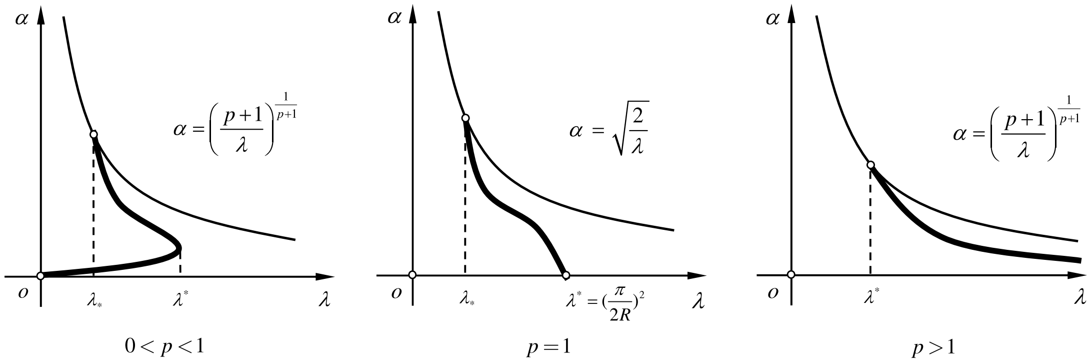

Fig. 1. Bifurcation diagrams for $f(u) = u^p$ in one dimension.

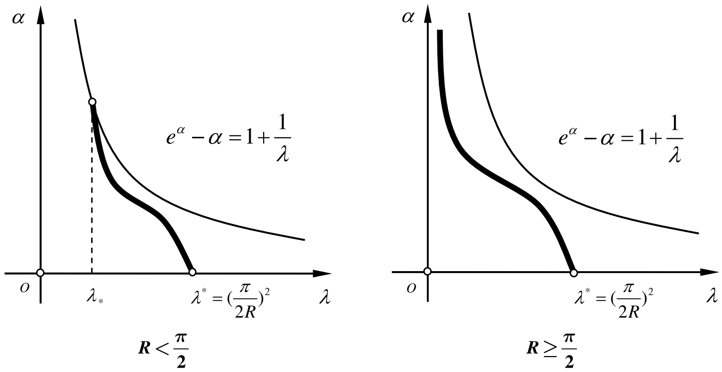

Fig. 2. Bifurcation diagrams for $f(u) = e^u - 1$ in one dimension.

Recall that nonexistence of solutions to (1.1) with $f(u) = u^p$ $(p \geqslant 1)$ in higher dimensional cases has been well studied. For $n = 2$ and $p = 1$, the problem comes from a model appearing in capillary surfaces and negative solutions describe pendent liquid drops of the equilibrium state (see Finn [12]). Wente [39] has proved the symmetry of negative solutions. Narukawa and Suzuki [23] have obtained nonexistence of negative solutions for small λ, and Finn [13] has derived a new proof for this fact by using a particular form of Green's Identity. For $n = 2$ and $p \geqslant 1$, a similar nonexistence result has previously been obtained by Ishimura [17]. For $n > 2$ and $p \geqslant \frac{n+2}{n-2}$, Ni and Serrin [25, Thm 3.4] have shown that the problem has no nontrivial radial solutions, it is also a corollary of the more general Pohozaev type identity by Pucci and Serrin [35]. For $\lambda = 1$, $n \geqslant 2$ and $f(u) = -\mu u + u^p$ $(\mu \geqslant 0, p > 1)$, Serrin [37, Thm 3] has shown that there exists a positive number R_1, depending only on p and n, such that the problem has no nontrivial positive solutions when $R < R_1$. For $n \geqslant 2$ and $f(u) = |u|^{p-1}u$ $(p \geqslant 1)$, Pan and Xing [30] have recently proved that no nontrivial (positive and sign-changing) solutions exist for small λ, they also consider the case $f(u) = e^u - 1$ and show that there exists some $R^* \in (0, \frac{\pi}{2})$ such that if $R < R^*$, no nontrivial positive solutions exist for small λ.

The purpose of this paper is to prove two more general nonexistence results about radial solutions of (1.1). Under suitable assumptions on f, we show that when λ is small, the problem with "superlinear" f has no nontrivial positive solutions, and when λ is large, the problem with "sublinear" f has no nontrivial positive solutions. If f is odd, then the former is also true for negative solutions and sign-changing solutions.

According to [36, Thm 8.2.2 and Remark], if $f(u)$ is locally Lipschitz continuous (only require upper Lipschitz continuity at $u = 0$), every positive $C^1(\overline{\mathcal{B}}_R)$ solution of (1.1) is radially symmetric about the origin. This result covers most of superlinear and sublinear nonlinearities f. For this reason and simplicity, we only consider radial solutions in this paper. By a solution u we mean that $u \in C^2(\overline{\mathcal{B}}_R)$ satisfies (1.1).

Denote $F(u) = \int_0^u f(t)dt$. Define $F(+\infty) = \lim_{u\to+\infty} F(u)$. We also introduce the condition:

$(f0)$ There exists some $\tilde{u} > 0$ such that $f(\tilde{u}) > 0$.

Our main results are the following.

Theorem 1.1 (*Superlinear Cases*)**.** *Let f be a continuous function with $f(0) = 0$. Assume that f satisfies $(f0)$ and the* superlinear *condition in the sense*

$$\frac{f(u)}{u} \quad \text{is increasing for } u > 0. \tag{1.2}$$