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A note on front tracking for the Keyfitz–Kranzer system

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ABSTRACT

A front tracking method is developed for the $n \times n$ symmetric Keyfitz–Kranzer system and convergence of the approximations to the strong generalized entropy solution of the system as defined by Panov [E.Y. Panov, On the theory of generalized entropy solutions of the Cauchy problem for a class of nonstrictly hyperbolic systems of conservation laws, Sb. Math. 191 (2000) 121–150] is proved. We also present numerical examples and compare the front tracking approximation with the approximations computed by two finite difference upwind schemes.

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1. Introduction

We consider the Cauchy problem for the $n \times n$ symmetric Keyfitz–Kranzer type system,

$$u_t + (u\phi(|u|))_x = 0, \quad (x,t) \in \mathbb{R} \times (0,T), \tag{1a}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{1b}$$

where $T > 0$ is given, $u = (u^{(1)}, \dots, u^{(n)}) : \mathbb{R} \times (0,T) \to \mathbb{R}^n$ the unknown and with $|u| := \sqrt{(u^{(1)})^2 + \cdots + (u^{(n)})^2}$, $u_0 = (u_0^{(1)}, \dots, u_0^{(n)}) \in L^\infty(\mathbb{R}, \mathbb{R}^n)$, the initial data, and $\phi(r) \in C^1(\mathbb{R}_+)$ a scalar function with

$$r\phi(r) \xrightarrow{r \to 0+} 0. \tag{2}$$

System (1) was first considered in [8,11] as a prototype of a nonstrictly hyperbolic system of conservation laws. In physics, it serves as a model for the elastic string [8], but it also appears in magnetohydrodynamics, where it is for example used to explain certain features of the solar wind [2]. Related systems of equations appear in chromatography [1,18,15] or in polymer flooding in porous media [16,14].

We denote the flux function $F(u) := u\phi(|u|)$. Its Jacobian matrix $A(u) := DF(u)$ is

$$A(u) = \phi(|u|)\,\mathbf{1} + \frac{\phi'(|u|)}{|u|} u \otimes u$$

where $\mathbf{1}$ denotes the $n \times n$ identity matrix. The matrix $A(u)$ is symmetric, therefore its eigenvalues are real and the corresponding collection of eigenvectors is complete, and system (1) is hyperbolic. The eigenvalues of $A(u)$ are $\lambda_1(u) = \phi(|u|) + \phi'(|u|)|u|$ with multiplicity 1 and $\lambda_2(u) = \phi(|u|)$ with multiplicity $n-1$. Due to the presence of eigenvalues with

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multiplicity >1, system (1) is not strictly hyperbolic in the sense of Lax [10]. The eigenspaces $E_i(u)$, $i = 1, 2$, corresponding to the eigenvalues $\lambda_i(u)$, are

$$E_1(u) = \operatorname{span}\{u\}, \qquad E_2(u) = E_1(u)^{\perp},$$

and thus we have for $v_i \in E_i$ with $|v_i| = 1$, denoting $r := |u|$,

$$\nabla\lambda_1(u) \cdot v_1 = 2\phi'(r) + \phi''(r)r, \qquad \nabla\lambda_2(u) \cdot v_2 = 0. \tag{3}$$

So the first characteristic field is either genuinely nonlinear or linearly degenerate (if $2\phi'(|u|) + \phi''(|u|)|u| = 0$) and the second characteristic field is always linearly degenerate.

Due to the nonlinearity of Eq. (1a), discontinuities can appear in its solution, no matter how smooth the initial data is. Therefore one seeks a *weak solution* to the equation, that is, one requires the differential equation to be satisfied only in the distributional sense,

$$\int_0^T \int_{\mathbb{R}} u\psi_t + u\phi(|u|)\psi_x \, dx\, dt + \int_{\mathbb{R}} u_0(x)\psi(x, 0)\, dx = 0, \quad \forall\, \psi \in C_0^{1,1}(\mathbb{R} \times [0, T)).$$

It is well known that weak solutions are not necessarily unique and therefore additional admissibility criteria have to be imposed to select the relevant solution. In the context of conservation laws, this is usually done by restricting to solutions satisfying in addition an entropy condition, which are therefore called *entropy solutions*. For system (1), the notion of an entropy solution was introduced by Freistühler [3,4] and by Panov [13]. It is defined as follows:

Definition 1.1 (*[13]*)**.** Let $\phi \in C(\mathbb{R}_+)$ satisfy (2). A bounded measurable vector-valued function is called a *strong generalized entropy solution* if the function $r(x, t) = |u(x, t)|$ is the entropy solution of

$$r_t + (\phi(r)r)_x = 0, \quad (x, t) \in \mathbb{R}, \tag{4a}$$

$$r(x, 0) = r_0(x) = |u_0(x)|, \quad x \in \mathbb{R}, \tag{4b}$$

that is, (4) is satisfied in the weak sense and in addition it holds for all entropy/entropy flux pairs (p, q), where $p(r)$ is convex and $q(r)$ defined by $q'(r) = (\phi(r)r)'p'(r)$,

$$p(r)_t + q(r)_x \le 0, \quad \text{in } \mathcal{D}'(\mathbb{R} \times (0, T)),$$

and u satisfies

$$\int_0^T \int_{\mathbb{R}} u\psi_t + u\phi(r)\psi_x \, dx\, dt + \int_{\mathbb{R}} u_0(x)\psi(x, 0)\, dx = 0, \quad \forall\, \psi \in C_0^{1,1}(\mathbb{R} \times [0, T)). \tag{5}$$

In [13], Panov proved existence and uniqueness of the entropy solution of (1):

Theorem 1.1. *There exists a unique strong generalized entropy solution $u \in L^\infty(\mathbb{R} \times (0, T))$ of (1) as in Definition 1.1. It can be obtained as the limit of solutions u^ϵ in $L^1_{loc}(\mathbb{R} \times (0, T))$ of the parabolic equation*

$$u_t^\epsilon + (\phi(|u^\epsilon|)u^\epsilon)_x = \epsilon u_{xx}^\epsilon$$

as $\epsilon \to 0$.

To prove that there is a unique u satisfying (5), in [12], the author defined $v := u/r$. Then v satisfies

$$(Av)_t + (Bv)_x = 0, \quad \text{in } \mathcal{D}'(\mathbb{R} \times (0, T)), \tag{6a}$$

$$v(x, 0) = v_0(x) = \frac{u_0(x)}{r_0(x)}, \quad x \in \mathbb{R}, \tag{6b}$$

$$A = A(x, t) = r(x, t), \quad (x, t) \in \mathbb{R} \times (0, T), \tag{6c}$$

$$B = B(x, t) = \phi(r(x, t))r(x, t), \quad (x, t) \in \mathbb{R} \times (0, T). \tag{6d}$$

For this type of equation, we have the following result:

Theorem 1.2 (*[12]*)**.** *Let v be a solution of*

$$(Av)_t + (Bv)_x = 0, \quad \text{in } \mathcal{D}'(\mathbb{R} \times (0, T)), \tag{7a}$$

$$v(x, 0) = v_0(x), \quad x \in \mathbb{R}, \tag{7b}$$

where $A, B \in L^\infty(\mathbb{R} \times (0, T))$ satisfy

$$\operatorname*{ess\,lim}_{t\to 0+} A(x, t) = A(x, 0) \quad \text{in } L^1_{loc}(\mathbb{R}),\ A(x, 0) \in L^\infty(\mathbb{R}); \tag{8a}$$

$$|B| \le N(\epsilon)(A + \epsilon) \quad \text{a.e. in } \mathbb{R} \times (0, T) \text{ for all } \epsilon > 0,\ \epsilon N(\epsilon) \xrightarrow{\epsilon\to 0+} 0; \tag{8b}$$

$$A_t + B_x = 0 \quad \text{in } \mathcal{D}'(\mathbb{R} \times (0, T)). \tag{8c}$$

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