J. Math. Anal. Appl. 407 (2013) 328–338

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

On a mixed problem with a nonlinear acoustic boundary condition for a non-locally reacting boundaries

A. Vicente [a,*], C.L. Frota [b]

[a] Universidade Estadual do Oeste do Paraná - CCET, Rua Universitária, 2069 - CEP 85819-110, Cascavel, PR, Brazil

[b] Universidade Estadual de Maringá - DMA, Av. Colombo, 5790 - CEP 87020-900, Maringá, PR, Brazil

ARTICLE INFO

Article history:
Received 3 January 2013
Available online 24 May 2013
Submitted by Xu Zhang

Keywords:
Nonlinear wave equation
Non-locally reacting boundary
Acoustic boundary conditions
Nonlinear boundary condition
Laplace–Beltrami operator

ABSTRACT

We discuss global solvability to the mixed problem for a nonlinear wave equation in domains with non-locally reacting boundaries and nonlinear acoustic boundary condition. We prove the existence and uniqueness of solution. Additionally, using the method of Nakao we demonstrate the decay of the associated energy (Stability).

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded and connected set with smooth boundary Γ, $n \geq 2$. Let Γ_1 be an open of Γ, $\partial\Gamma_1$ the smooth boundary of Γ_1 and $\Gamma_0 = \Gamma \setminus \Gamma_1$, Γ_0 and Γ_1 with positive measure. Then $\overline{\Gamma_1}$ is a sub-manifold of Γ and $\overline{\Gamma_1} = \Gamma_1 \cup \partial\Gamma_1$ is a compact, C^∞, manifold with boundary. In this paper, we are concerned with the existence and uniqueness of global solutions and the decay of the associated energy to the following nonlinear problem

$$u'' - \Delta u + \varphi(u') = 0 \quad \text{in } \Omega \times (0, \infty); \tag{1.1}$$

$$u = 0 \quad \text{on } \Gamma_0 \times (0, \infty); \tag{1.2}$$

$$\frac{\partial u}{\partial \nu} = \frac{\delta'}{M\left(\int_{\Gamma_1} |\delta|^2 d\Gamma\right)} \quad \text{on } \Gamma_1 \times (0, \infty); \tag{1.3}$$

$$f\delta'' - M\left(\int_{\Gamma_1} |\delta|^2 d\Gamma\right)(\Delta_\Gamma \delta - \delta) + \psi(\delta') = -u' \quad \text{on } \Gamma_1 \times (0, \infty); \tag{1.4}$$

$$\delta = 0 \quad \text{on } \partial\Gamma_1 \times (0, \infty); \tag{1.5}$$

$$u(x, 0) = u_0(x), \qquad u'(x, 0) = u_1(x) \quad \text{for } x \in \Omega; \tag{1.6}$$

$$\delta(x, 0) = \delta_0(x), \qquad \delta'(x, 0) = M\left(\int_{\Gamma_1} |\delta_0|^2 d\Gamma\right) \frac{\partial u_0}{\partial \nu}(x) \quad \text{for } x \in \Gamma_1, \tag{1.7}$$

* Corresponding author.
E-mail addresses: andre.vicente@unioeste.br, avicentemath@gmail.com (A. Vicente), clfrota@uem.br (C.L. Frota).

0022-247X/$ – see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.jmaa.2013.05.054

where $' = \frac{\partial}{\partial t}$; $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ and Δ_Γ are the Laplace and the Laplace–Beltrami operators, respectively; ν is the outward normal unit vector on Γ_1; $f : \overline{\Gamma_1} \to \mathbb{R}$, $\varphi : \mathbb{R} \to \mathbb{R}$, $\psi : \mathbb{R} \to \mathbb{R}$, $M : [0, \infty) \to \mathbb{R}$, $u_0, u_1 : \Omega \to \mathbb{R}$ and $\delta_0 : \Gamma_1 \to \mathbb{R}$ are given functions.

Recently Frota, Medeiros and Vicente [20] studied the nonlinear equation

$$u'' - \Psi\left(\int_\Omega u^2 dx\right)\Delta u + \alpha u' + \beta |u'|^\lambda u' = 0 \quad \text{in } \Omega \times (0, \infty), \tag{1.8}$$

with boundary conditions

$$u = 0 \quad \text{on } \Gamma_0 \times (0, \infty), \tag{1.9}$$

$$\frac{\partial u}{\partial \nu} = \delta' \quad \text{on } \Gamma_1 \times (0, \infty), \tag{1.10}$$

$$f\delta'' - c^2 \Delta_\Gamma \delta + g\delta + h\delta' = -u' \quad \text{on } \Gamma_1 \times (0, \infty), \tag{1.11}$$

and initial data

$$u(x, 0) = u_0(x), \qquad u'(x, 0) = u_1(x) \quad \text{for } x \in \Omega; \tag{1.12}$$

$$\delta(x, 0) = \delta_0(x), \qquad \delta'(x, 0) = \frac{\partial u_0}{\partial \nu}(x) \quad \text{for } x \in \Gamma_1, \tag{1.13}$$

where $\Omega \subset \mathbb{R}^n$, $n = 2, 3$; f, g and h are given real functions on Γ_1; α, β and c are positive constants; $\lambda > 1$, if $n = 2$ and $1 < \lambda \le 2$, if $n = 3$. The authors proved the existence and uniqueness of global solution and uniform decay rates of the energy when $t \to \infty$ and the linear term $\alpha u'$ (in (1.8)) was essential because it worked in connection with the nonlinear term $\beta|u'|^\lambda u'$. Moreover the authors showed local existence of solution to (1.8)–(1.13) when $\alpha = \beta = 0$ and a nonlinear equation was considered instead of (1.10). When $\beta = 0$ the problem (1.8)–(1.13) is more difficult and it was investigated by Vicente and Frota [35]. In this case, since the dissipation in Ω is weaker it was necessary an assumption of small initial data in order to prove the existence, uniqueness of solution and the decay of the energy.

On the other hand, $|u'|^\lambda u'$ is a classical nonlinear dissipative term studied by Lions [26] and it has been considered by many authors, see [11,15,17,19,29,36]. We can cite the following papers: Cavalcanti, Domingos Cavalcanti, Prates Filho and Soriano [11], Liu and Yu [29] and Vitillaro [36] where this dissipative term was considered over the boundary and u' was treated as the trace of a function defined in Ω. Let us also note that stabilization to the wave equations have been studied by many authors [1,24,25,28,30].

The boundary conditions (1.10)–(1.11) are called *acoustic boundary conditions to non-locally reacting boundary* and its formulation is associated with the study of acoustic wave motion. Precisely, we suppose that a fluid is filled in Ω and it is at rest except for acoustic wave motion and the surface Γ_1 act like an elastic membrane to the excess pressure of acoustic wave. Then the normal displacement $\delta(x, t)$ of the point x in the boundary at time t satisfies the Eq. (1.11). The Eq. (1.10) is an impenetrability condition and it is obtained from the continuity of velocity at the boundary. When $c = 0$ the boundary conditions (1.10)–(1.11) are the classical acoustic boundary conditions introduced by Beale and Rosencrans [3]. In this case, there exist many contributions on the existence, uniqueness and stability, see [16,18,19,21,22,32,34] and references therein. We emphasize Graber [21,22] and J. Y. Park and S. H. Park [32] where the authors studied problems with nonlinear impenetrability condition. See also [10,7–9,12] which contain very interesting related results.

The nonlinear equation (1.4) is a boundary version of the nonlinear Carrier equation which was introduced in 1945 by Carrier [6] and it is an alternative model for small vibrations of an elastic string with fixed ends when the changes in tension are not small. It is important to explain that, with the classical acoustic boundary conditions of Beale and Rosencrans, the estimates to δ are not sufficient to consider the term $M(\int_{\Gamma_1} \delta^2 d\Gamma)$ and the Eq. (1.4) cannot be considered.

The purpose of this paper is to prove the existence and uniqueness of global solution and the decay of the energy to (1.1)–(1.7). To the best of our knowledge this is first paper with nonlinear acoustic boundary condition for non-locally reacting boundaries which is more interesting in physical situations. Moreover, taking into account that (1.3) is a nonlinear equation, we also obtain some advances on the impenetrability condition. The assumptions we have made here allow us to include the dissipative nonlinear term of (1.8) which was discussed in [20], but now over the boundary, i.e., we can take $\psi(\delta') = \alpha\delta' + \beta|\delta'|^\sigma\delta'$ and when $n = 3$ we get more liberty on σ. We note that under the assumption on function M we cannot proceed (in estimate I) as in [20] and the alternative way to get our result is to consider an appropriate assumption on the size of initial data, as in Brito [5]. Finally, using the ideas of Nakao [31] we prove the stability result.

Our paper is organized as follows: in Section 2 we give the notations and assumptions. In Section 3 we prove the existence and uniqueness of solution. The stability is studied in Section 4.