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Dual univariate m-ary subdivision schemes of de Rham-type

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ABSTRACT

In this paper, we present an algebraic perspective of the de Rham transform of a binary subdivision scheme and propose an elegant strategy for constructing dual m-ary approximating subdivision schemes of de Rham-type, starting from two primal schemes of arity m and 2, respectively. On the one hand, this new strategy allows us to show that several existing dual corner-cutting subdivision schemes fit into a unified framework. On the other hand, the proposed strategy provides a straightforward algorithm for constructing new dual subdivision schemes having higher smoothness and higher polynomial reproduction capabilities with respect to the two given primal schemes.

1. Introduction

Univariate *subdivision schemes* are iterative methods for representing smooth curves via the specification of a coarse polygon $\mathbf{f}^{(0)} := \{f_i^{(0)},\ i \in \mathbb{Z}\}$ and a set of refinement rules mapping the sequence of points $\mathbf{f}^{(k)}$, $k \geq 0$, into the denser sequence of points $\mathbf{f}^{(k+1)}$. If the k-th refinement step consists of m refinement rules, the subdivision scheme is said to be of *arity* m and, if the refinement rules are linear combinations of the coarser points, the subdivision scheme is called *linear*. Let a_i, $i \in \mathbb{Z}$, be the coefficients appearing in the linear combination. Then, for each $k \geq 0$ the m refinement rules read as

$$f_{mi+\ell}^{(k+1)} := \sum_{j\in\mathbb{Z}} a_{m(i-j)+\ell} f_j^{(k)}, \quad \ell = 0, \ldots, m-1.$$

The set of coefficients $\{a_i,\ i \in \mathbb{Z}\}$ is called *subdivision mask* and is denoted by $\mathbf{a}$. The associated subdivision scheme is denoted by $S_{\mathbf{a}}$ and can be equivalently seen as the repeated application of the subdivision matrix $S = \{s(i,j) = a_{i-mj} : i,j \in \mathbb{Z}\}$. To establish a notion of convergence, we associate to the sequence of refined data $\{\mathbf{f}^{(k)},\ k \geq 0\}$ a sequence of *parameter values* $\mathbf{t}^{(k)} = \{t_i^{(k)},\ i \in \mathbb{Z}\}$ with $t_{i+1}^{(k)} - t_i^{(k)} = m^{-k}$, and we define the piecewise linear function $F^{(k)}$ that interpolates the data $\mathbf{f}^{(k)}$ at the parameters $\mathbf{t}^{(k)}$. If, for every initial data $\mathbf{f}^{(0)}$, the sequence $\{F^{(k)},\ k \geq 0\}$ is convergent to a continuous function $g_{\mathbf{f}^{(0)}}$, then the subdivision scheme is said to be C^0-*convergent*. Moreover, the scheme is said to be C^r-*convergent* if $g_{\mathbf{f}^{(0)}}$ is a $C^r(\mathbb{R})$ function. In this paper, we only consider subdivision schemes that are convergent and *non-singular*, so that $g_{\mathbf{f}^{(0)}} = 0$ if and only if $\mathbf{f}^{(0)} = 0$. Assuming that the *support* of the subdivision mask $\mathbf{a}$ is $[0, N] \cap \mathbb{Z}$ (*i.e.* $a_i = 0$ for $i < 0$ and $i > N$ as well as $a_0, a_N \neq 0$), then the support of the limit function is given by $\left[0, \frac{N}{m-1}\right]$ (see for example [8, Section 1]).

Support size and smoothness are considered as mutually conflicting properties of a subdivision scheme because a high degree of smoothness generally requires a large support, thus leading to a more global influence of each initial data value on the limit function. Raising the arity of the subdivision scheme provides a way to overcome this dilemma to some extents. For

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example, the ternary and quaternary approximating 4-point schemes discussed in [20] and [22], respectively, have smaller support and higher smoothness than the binary approximating 4-point scheme in [14], and all three schemes reproduce cubic polynomials by construction. The latter means that, whenever starting from data on a cubic polynomial, their limits are exactly that cubic polynomial. So, subdivision schemes of arity $m > 2$, although much less known in the literature, are potentially more useful because smoother but with smaller support than their binary counterparts.

Another way of increasing the smoothness of a subdivision scheme in exchange of a slight increase of its support width is to use *dual* subdivision schemes instead of *primal* ones. From a geometric point of view, primal schemes are those that retain or modify the old vertices and create $m-1$ new vertices at each old edge of the control polygon. Dual schemes, instead, create m new points at the old edges and discard the old vertices. The importance of dual schemes in practical applications is due to the fact that they can be smoother than the primal schemes having the same degree of polynomial reproduction, in exchange of a slight increase of the support width. This is the case, for instance, of the dual C^2 four-point subdivision scheme presented in [14]. In fact, the support width of its subdivision mask is increased only by one with respect to that of the interpolatory (primal) C^1 four-point scheme in [15,17], and they both reproduce cubic polynomials.

The first univariate, linear subdivision scheme appeared in the literature is the arity-2 (binary) scheme having mask

$$\mathbf{a} = [w, 1-w, 1-w, w] \quad \text{with } w \in \left(0, \frac{1}{2}\right). \tag{1}$$

The particular choice of $w = \frac{1}{3}$ corresponds to the de Rham scheme [11] and that of $w = \frac{1}{4}$ to the Chaikin's scheme [1]. Since the refinement rules associated to (1) are

$$f_{2i}^{(k+1)} = (1-w)f_i^{(k)} + wf_{i+1}^{(k)}, \qquad f_{2i+1}^{(k+1)} = wf_i^{(k)} + (1-w)f_{i+1}^{(k)}, \quad k \geq 0, i \in \mathbb{Z},$$

the $(k+1)$-level new vertices $f_{2i}^{(k+1)}$ and $f_{2i+1}^{(k+1)}$ are constructed at points w and $1-w$ of the way along each edge of the k-level control polygon and so each line segment connecting $f_i^{(k)}$ and $f_{i+1}^{(k)}$ is partitioned with the ratio $w : (1-2w) : w$. The mask in (1) thus provides a family of 2-point corner-cutting subdivision schemes.

In the recent paper [12], Dubuc observed that a single step of the binary subdivision scheme having mask (1) with $w = \frac{1}{4}$ can be seen as the subsequent application of two steps of the linear binary B-spline scheme with mask $[\frac{1}{2}, 1, \frac{1}{2}]$, followed by the selection of the obtained points with odd index only (see Fig. 1, first row). Prompted by this observation, Dubuc also defined the *de Rham transform* of a binary subdivision scheme with matrix $S = \{s(i,j) : i,j \in \mathbb{Z}\}$ as the subdivision scheme with matrix $\tilde{S} = \{s_2(2i+1,j) : i,j \in \mathbb{Z}\}$, where $s_2(i,j)$ are the entries of S^2. He observed that, for any binary subdivision scheme, we can define its de Rham transform which generalizes the de Rham and Chaikin corner cutting. In particular, in [12] he applies the de Rham transform to three families of interpolatory subdivision schemes and shows that, although the interpolatory property is lost, the de Rham transform can provide subdivision schemes that are smoother than the original ones. In [13], this idea was originally applied to binary interpolatory Hermite subdivision schemes in order to define new smoother non-interpolatory Hermite schemes.

Exploiting the notion of *subdivision symbol*, in this paper we provide an algebraic perspective of the de Rham transform. More precisely, denoted by $a(z) = \sum_{i\in\mathbb{Z}} a_i z^i,\ z \in \mathbb{C} \setminus \{0\}$ the symbol of the binary subdivision scheme $S_{\mathbf{a}}$ with mask $\mathbf{a} = \{a_i,\ i \in \mathbb{Z}\}$, we can show that the symbol of the de Rham transform of $S_{\mathbf{a}}$ is the product of two special Laurent polynomial factors. In fact, for a given sequence of vertices $\mathbf{f}^{(k)} = \{f_i^{(k)},\ i \in \mathbb{Z}\}$, the double application of the subdivision operator $S_{\mathbf{a}}$ provides the points $\tilde{f}_i = \sum_{j\in\mathbb{Z}} a_{i-2j} \left(\sum_{\ell\in\mathbb{Z}} a_{j-2\ell} f_\ell^{(k)}\right), i \in \mathbb{Z}$. Thus, defining the sequence of points at level $k+1$ as the subset of the odd-indexed points $\tilde{f}_{2i+1}$, we get

$$\mathbf{f}^{(k+1)} = \left\{ f_i^{(k+1)} = \sum_{\ell\in\mathbb{Z}} \left(\sum_{j\in\mathbb{Z}} a_{2i+1-4\ell-2j}\, a_j \right) f_\ell^{(k)}, \quad i \in \mathbb{Z} \right\}.$$

Hence, denoting by $S_{\mathbf{c}}$ the de Rham transform of $S_{\mathbf{a}}$, i.e. the subdivision scheme mapping $\mathbf{f}^{(k)}$ into $\mathbf{f}^{(k+1)}$, it turns out that the subdivision mask of $S_{\mathbf{c}}$ is given by $\mathbf{c} = \left\{c_i = \sum_{j\in\mathbb{Z}} a_{2i+1-2j}\, a_j,\ i \in \mathbb{Z}\right\}$ and its symbol is

$$c(z) = \sum_{i\in\mathbb{Z}} c_i z^i = \sum_{r\in\mathbb{Z}} a_{2r+1} z^r \sum_{j\in\mathbb{Z}} a_j z^j = a_{odd}(z)a(z).$$

This algebraic interpretation of the de Rham transform allows us to extend the results in [12] in two different directions: firstly, we replace the double step of a primal binary scheme by the subsequent application of two different primal binary schemes; secondly, we allow one of the two primal schemes to be of arity different from 2. The so obtained extended de Rham-type strategy provides a corner cutting scheme of arbitrary arity. For example, the ternary scheme in [23, pag.53] (known as the "neither" scheme), is a corner cutting scheme which can be interpreted as a de Rham-type scheme obtained by applying first a linear ternary B-spline scheme, then a linear binary B-spline scheme, both followed by a selection of the odd entries only (see Fig. 1, second row).

In this paper, we will show that any de Rham-type m-ary subdivision scheme is *dual* by construction and each subdivision step of such a scheme can be obtained by applying to the k-level vertices first a primal m-ary subdivision operator $S_{\mathbf{b}}$, then