J. Math. Anal. Appl. 400 (2013) 541–557

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Strong duality for generalized monotropic programming in infinite dimensions

Regina S. Burachik, Saba N. Majeed *

School of Mathematics and Statistics, University of South Australia, Mawson Lakes, SA 5095, Australia

ARTICLE INFO

Article history:
Received 7 June 2012
Available online 6 November 2012
Submitted by Heinz Bauschke

Keywords:
Monotropic programming
Convex functions
Convex cones
Fenchel duality
Epigraph of a convex function
Conjugate function

ABSTRACT

We establish duality results for the generalized monotropic programming problem in separated locally convex spaces. We formulate the generalized monotropic programming (GMP) as the minimization of a (possibly infinite) sum of separable proper convex functions, restricted to a closed and convex cone. We obtain strong duality under a constraint qualification based on the closedness of the sum of the epigraphs of the conjugates of the convex functions. When the objective function is the sum of finitely many proper closed convex functions, we consider two types of constraint qualifications, both of which extend those introduced in the literature. The first constraint qualification ensures strong duality, and is equivalent to the one introduced by Boţ and Wanka. The second constraint qualification is an extension of Bertsekas' constraint qualification and we use it to prove zero duality gap.

1. Introduction

The Monotropic Programming (MP) problem was introduced by Rockafellar in [1] and has been widely studied (cf. [2–4]). In its classical form, MP is defined in a finite dimensional setting and it involves minimizing a finite sum of proper and convex separable functions restricted to a closed subspace. We focus on a generalization of this problem, the generalized monotropic programming (GMP), which consists of minimizing a (finite or infinite) sum of proper and convex functions defined on (possibly different) locally convex spaces. To define the problem, denote $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ and let I be an arbitrary index set. Consider a family of real separated locally convex spaces $\{X_i\}_{i\in I}$, and a family $\{f_i\}_{i\in I}$ of proper and convex functions such that $f_i : X_i \to \overline{\mathbb{R}}$ for all $i \in I$. Take $X := \prod_{i\in I} X_i$ and consider the sum of $\{f_i\}_{i\in I}$, defined as $f : X \to \overline{\mathbb{R}}$ such that $f(x) := \sum_{i\in I} f_i(x_i)$. The meaning of the right-hand side of the last expression in the case in which I is infinite is recalled later on in Definition 2.2. The GMP problem we study is as follows:

$$\begin{aligned} &\min \sum_{i\in I} f_i(x_i) \\ &\text{subject to } x \in K, \end{aligned} \tag{P}$$

where $x_i \in X_i$ for all $i \in I$, and the constraint set K is a closed and convex cone contained in $X = \prod_{i\in I} X_i$. As far as we know, the only work dealing with an infinite sum of convex functions is [5]. The functions in [5] are defined in Banach spaces, however, the definition and the properties of the infinite sums mentioned in [5] can be stated for functions defined on separated locally convex spaces.

Following [6,3], we say that strong duality holds when the optimal primal and dual values coincide, and the dual value is attained. If we only have equality of the primal and dual optimal values, we say that we have zero duality gap. Rockafellar

* Corresponding author.
E-mail addresses: regina.burachik@unisa.edu.au (R.S. Burachik), saba.majeed@mymail.unisa.edu.au, majsy001@mymail.unisa.edu.au (S.N. Majeed).

0022-247X/$ – see front matter © 2012 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2012.10.052

[1,4] was the first to use a variant of the ϵ-descent method to prove zero duality gap for the MP problem. More recently, Bertsekas [7] has modified Rockafellar's method and applied it for solving the extended MP problem. The latter problem has for the objective function a finite sum of extended real-valued functions which can have domains in different finite dimensional spaces [7], and use a subspace S as a constraint set. To obtain zero duality gap in this context, Bertsekas used projections on an outer approximation of the ϵ-subdifferential and used a constraint qualification involving the closedness of the Minkowski sum of ϵ-subdifferentials. Bertsekas' constraint qualification (cf. [7, Proposition 4.1]) requires that the set

$$A_\epsilon(x) = \partial_\epsilon \delta_S(x) + \partial_\epsilon \bar{f}_1(x) + \cdots + \partial_\epsilon \bar{f}_m(x) \tag{1}$$

is closed for all feasible solutions $x = (x_1, \ldots, x_m)$ and every $\epsilon > 0$, where $\bar{f}_i(x) := f_i(x_i)$ for each $i = 1, \ldots, m$ (for the definition of $\bar{f}_i$, see Remark 3.1(1)).

In locally convex spaces, Boţ and Csetnek [3] proved zero duality gap for the extended MP problem under alternative assumptions. Boţ and Csetnek used in [3] an extension to separated locally convex spaces of Bertsekas' constraint qualification (1). Our purpose is to study strong duality for our general version GMP of the MP. We obtain strong duality under new constraint qualifications (see Theorems 3.4 and 3.5). In Theorem 3.4 we prove that, when I is finite and the constraint set K is nonempty closed and convex, strong duality for the Problem (P) holds if the set

$$epi\delta_K^* + epi\bar{f}_1^* + \cdots + epi\bar{f}_m^* \tag{2}$$

is $weak^*$ closed. The $epi\bar{f}_i^*$ is the epigraph of the conjugate function of $\bar{f}_i$ defined above. We also show, in Theorem 3.4, that the above constraint qualification is equivalent to the ones used in [8, p. 2798], [6, Theorem 3.2.6] for the case of locally convex spaces and [9, Corollary 3] for the case of Banach spaces, to obtain generalized Fenchel's duality. Namely, it is equivalent to the $weak^*$ closedness of the set $epif^* + epig^*$ in case $f(x)$ is defined as in Problem (P) and $g(x) = \delta_K(x)$. Still for the finite sum, we use in Theorem 3.6, an extension of Bertsekas' constraint qualification (1) to obtain zero duality gap. This constraint qualification requires that the set

$$A_\epsilon(x) = \partial_\epsilon \delta_K(x) + \partial_\epsilon \bar{f}_1(x) + \cdots + \partial_\epsilon \bar{f}_m(x) \tag{3}$$

is $weak^*$ closed for all feasible solutions $x = (x_1, \ldots, x_m)$ and every $\epsilon > 0$, where $\partial_\epsilon \bar{f}_i(x)$ is the ϵ-subdifferential of $\bar{f}_i$ at x. Note that the constraint subspace S used in (1) has been replaced in (3) by any closed convex cone K.

Theorem 3.5 considers the case of I infinite and an arbitrary closed and convex constraint set K. For this case we prove strong duality when

$$epi\delta_K^* + \sum_{i\in I}^{*} epi\bar{f}_i^* \tag{4a}$$

is $weak^*$ closed, and

$$epif^* = \sum_{i\in I}^{*} epi\bar{f}_i^*. \tag{4b}$$

This constraint qualification, which is new in the literature, involves the $weak^*$ closedness of the sum of the epigraphs of the conjugate functions and an additional condition on the summability of the epigraph of the conjugate of the infinite sum. Corollary 3.1 describes conditions under which the constraint qualification (4a) is enough to ensure strong duality.

The outline of the paper is as follows. In Section 2, we review the necessary definitions and preliminary results. In Section 3, we introduce the GMP problem and its dual in a separated locally convex space and introduce our new constraint qualifications (2) and (4a)–(4b) to obtain strong duality for the GMP problem. Still in Section 3, we introduce the constraint qualification (3) to show zero duality gap which generalizes the one used in [7]. We end Section 3 with an example illustrating the fact that our new constraint qualification (2) is not weaker than (3). Section 4 contains our conclusions. Some needed technical facts from real analysis are proved in the Appendix.

2. Preliminaries

We collect in this section some definitions and properties from convex analysis which can be found e.g., in [6,10,11]. Let X denote a locally convex space, and X^* its topological dual space endowed with the $weak^*$ topology $w^*(X^*, X)$. If D is a subset of X^*, the *$weak^*$ closure* of D will be denoted by $\overline{D}^{w^*}$. Let C be a non empty subset of X. The *indicator function* associated with the set C, $\delta_C : X \to \overline{\mathbb{R}}$ is defined by

$$\delta_C(x) := \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{otherwise.} \end{cases}$$

The *support function* $\sigma_C : X^* \to \overline{\mathbb{R}}$ is defined by $\sigma_C(v) = \sup\{\langle v, x\rangle : x \in C\}$, where $\langle \cdot, \cdot \rangle$ is the *duality product* in $X^* \times X$. Recall that, given $\epsilon \geq 0$, the *ϵ-normal set* of C, which is denoted by $N_C^\epsilon(x)$, is defined as

$$N_C^\epsilon(x) := \begin{cases} \{x^* \in X^* : \langle x^*, y - x\rangle \leq \epsilon, \forall y \in C\} & \text{if } x \in C, \\ \phi & \text{otherwise.} \end{cases} \tag{4}$$